

# Preserving Randomness for Adaptive Algorithms

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## Abstract

We introduce the concept of a *randomness steward*, a tool for saving random bits when executing a randomized estimation algorithm **Est** on many adaptively chosen inputs. For each execution, the chosen input to **Est** remains hidden from the steward, but the steward chooses the randomness of **Est** and, crucially, is allowed to modify the output of **Est**. The notion of a steward is inspired by *adaptive data analysis*, introduced by Dwork et al. [DFH<sup>+</sup>15c]. Suppose **Est** outputs values in  $\mathbb{R}^d$ , has  $\ell_\infty$  error  $\varepsilon$ , has failure probability  $\delta$ , uses  $n$  coins, and will be executed  $k$  times. For any  $\gamma > 0$ , we construct a computationally efficient steward with  $\ell_\infty$  error  $O(\varepsilon d)$ , failure probability  $k\delta + \gamma$ , and randomness complexity  $n + O(k \log(d+1) + (\log k) \log(1/\gamma))$ . To achieve these parameters, the steward uses a pseudorandom generator for what we call the *block decision tree* model, combined with a scheme for shifting and rounding the outputs of **Est**. (The generator is a variant of the [INW94] generator for space-bounded computation.) We also give variant stewards that achieve tradeoffs in parameters.

As applications of our steward, we give time- and randomness-efficient algorithms for estimating the acceptance probabilities of many adaptively chosen Boolean circuits and for simulating any algorithm with an oracle for **promise-BPP** or **APP**. We also give a randomness-efficient version of the Goldreich-Levin algorithm; our algorithm finds all Fourier coefficients with absolute value at least  $\theta$  of a function  $F : \{0, 1\}^n \rightarrow \{-1, 1\}$  using  $O(n \log n) \cdot \text{poly}(1/\theta)$  queries to  $F$  and  $O(n)$  random bits, improving previous work by Bshouty et al. [BJT99]. Finally, we prove a randomness complexity lower bound of  $n + \Omega(k) - \log(\delta'/\delta)$  for any steward with failure probability  $\delta'$ , which nearly matches our upper bounds in the case  $d \leq O(1)$ .

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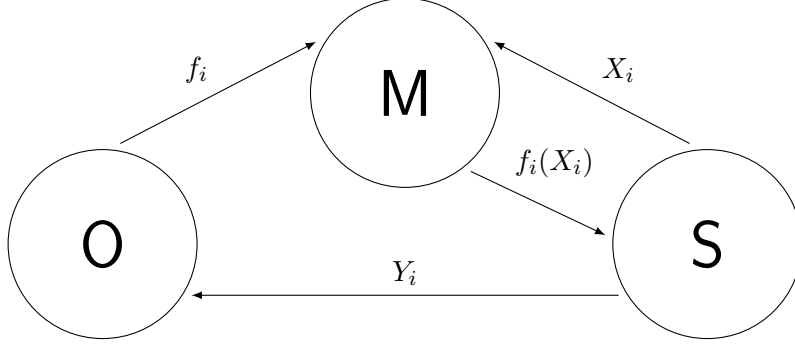


Figure 1: A single round of interaction in the definition of a steward. Here,  $f_i(X) \stackrel{\text{def}}{=} \text{Est}(C_i, X)$ .

## 1 Introduction

Let  $\text{Est}$  be a randomized algorithm that estimates some quantity  $\mu(C) \in \mathbb{R}^d$  when given input  $C$ . The canonical example is the case when  $C$  is a Boolean circuit,  $d = 1$ ,  $\mu(C) \stackrel{\text{def}}{=} \Pr_x[C(x) = 1]$ , and  $\text{Est}$  estimates  $\mu(C)$  by evaluating  $C$  at several randomly chosen points. Suppose that  $\text{Est}$  uses  $n$  random bits, and  $\Pr[\|\text{Est}(C) - \mu(C)\|_\infty > \varepsilon] \leq \delta$ .

Furthermore, suppose we want to use  $\text{Est}$  as a *subroutine*, executing it on inputs  $C_1, C_2, \dots, C_k$ , where each  $C_i$  is chosen adaptively based on the previous outputs of  $\text{Est}$ . The naïve implementation uses  $nk$  random bits and fails with probability at most  $k\delta$ .

In this work, we show how to generically improve the randomness complexity of any algorithm with this structure, without increasing the number of executions of  $\text{Est}$ , at the expense of mild increases in the error and failure probability. Our algorithm efficiently finds  $Y_1, \dots, Y_k \in \mathbb{R}^d$  with  $\|Y_i - \mu(C_i)\|_\infty \leq O(\varepsilon d)$  for every  $i$ , our algorithm has failure probability  $k\delta + \gamma$  for any  $\gamma > 0$ , and our algorithm uses a total of  $n + O(k \log(d+1) + (\log k) \log(1/\gamma))$  random bits.

### 1.1 The randomness steward model

A simple but important observation is that an algorithm that uses  $\text{Est}$  as a subroutine only sees the *outputs* of  $\text{Est}$ , not the coin tosses of  $\text{Est}$ . To make this fact explicit, we introduce the concept of a *randomness steward*. We first describe the model informally.

Imagine that  $O$ , the *owner*, plans to execute  $\text{Est}$  on  $k$  adaptively chosen inputs. To improve randomness efficiency, she hires a *steward*  $S$  and gives  $S$  “stewardship” over her random bits. When  $O$  wants to execute  $\text{Est}$  on an input  $C_i$ , she delegates the task to  $S$ , who chooses a randomness string  $X_i \in \{0, 1\}^n$ . To prevent  $O$  from seeing  $X_i$ ,  $O$  is required to give  $C_i$  to a *mediator*  $M$ , who evaluates  $\text{Est}(C_i, X_i)$  on behalf of  $S$ . Based on the output of  $\text{Est}$ ,  $S$  chooses a value  $Y_i \in \mathbb{R}^d$  to return to  $O$ . The steward  $S$  judiciously chooses these  $X_i, Y_i$  values so as to “spend” as little randomness as possible. The requirement is that with high probability,  $Y_i$  is close to  $\mu(C_i)$  for every  $i$ .

Now, we describe the model rigorously. Say that a function  $f : \{0, 1\}^n \rightarrow \mathbb{R}^d$  is  $(\varepsilon, \delta)$ -concentrated at  $\mu \in \mathbb{R}^d$  if  $\Pr_{X \in \{0, 1\}^n}[\|f(X) - \mu\|_\infty > \varepsilon] \leq \delta$ . In each round  $i$ , the chosen input  $C_i$  defines a concentrated function  $f_i(X) \stackrel{\text{def}}{=} \text{Est}(C_i, X)$ , so it is equivalent to imagine that  $O$  picks an arbitrary concentrated function. (See Figure 1.) In the following definition,  $\varepsilon'$  is the error of the steward, and  $\delta'$  is its failure probability.

**Definition 1.** An  $(\varepsilon', \delta')$ -steward for  $k$  adaptively chosen  $(\varepsilon, \delta)$ -concentrated functions  $f_1, \dots, f_k : \{0, 1\}^n \rightarrow \mathbb{R}^d$  is a randomized algorithm  $S$  that interacts with an owner  $O$  through a mediator  $M$  according to the following protocol.

1. For  $i = 1$  to  $k$ :
  - (a)  $O$  chooses  $f_i : \{0, 1\}^n \rightarrow \mathbb{R}^d$  that is  $(\varepsilon, \delta)$ -concentrated at some point  $\mu_i \in \mathbb{R}^d$  and gives it to  $M$ .
  - (b)  $S$  chooses  $X_i \in \{0, 1\}^n$  and gives it to  $M$ .
  - (c)  $M$  gives  $f_i(X_i) \in \mathbb{R}^d$  to  $S$ .
  - (d)  $S$  chooses  $Y_i \in \mathbb{R}^d$  and gives it to  $O$ .

Write  $O \leftrightarrow S$  (“the interaction of  $O$  with  $S$ ”) to denote the above interaction. The requirement on  $S$  is that for all  $O$ ,

$$\Pr[\max_i \|Y_i - \mu_i\|_\infty > \varepsilon' \text{ in } O \leftrightarrow S] \leq \delta'.$$

The probability is taken over the internal randomness of  $S$  and  $O$ .

## 1.2 Our results

### 1.2.1 Main result: A steward with good parameters

Our main result is the explicit construction of a steward that simultaneously achieves low error, low failure probability, and low randomness complexity:

**Theorem 1.** *For any  $n, k, d \in \mathbb{N}$  and any  $\varepsilon, \delta, \gamma > 0$ , there exists an  $(O(\varepsilon d), k\delta + \gamma)$ -steward for  $k$  adaptively chosen  $(\varepsilon, \delta)$ -concentrated functions  $f_1, \dots, f_k : \{0, 1\}^n \rightarrow \mathbb{R}^d$  with randomness complexity*

$$n + O(k \log(d + 1) + (\log k) \log(1/\gamma)).$$

*The total running time of the steward is  $\text{poly}(n, k, d, \log(1/\varepsilon), \log(1/\gamma))$ .*

We also give several variant stewards that achieve tradeoffs in parameters. (See Figure 2.)

### 1.2.2 Application: Acceptance probabilities of Boolean circuits

Our first concrete application of Theorem 1 is a time- and randomness-efficient algorithm for estimating the acceptance probabilities of many adaptively chosen Boolean circuits.

**Corollary 1.** *There exists a randomized algorithm with the following properties. Initially, the algorithm is given parameters  $n, k \in \mathbb{N}$  and  $\varepsilon, \delta > 0$ . Then, in round  $i$  ( $1 \leq i \leq k$ ), the algorithm is given a Boolean circuit  $C_i$  on  $n$  input bits and outputs a number  $Y_i \in [0, 1]$ . Here,  $C_i$  may be chosen adversarially based on  $Y_1, \dots, Y_{i-1}$ . With probability  $1 - \delta$ , every  $Y_i$  is  $\mu(C_i) \pm \varepsilon$ , where  $\mu(C_i) \stackrel{\text{def}}{=} \Pr_x[C_i(x) = 1]$ . The total running time of the algorithm is*

$$O\left(\frac{\log k + \log(1/\delta)}{\varepsilon^2} \cdot \sum_{i=1}^k \text{size}(C_i)\right) + \text{poly}(n, k, 1/\varepsilon, \log(1/\delta)),$$

*and the total number of random bits used by the algorithm is  $n + O(k + (\log k) \cdot \log(1/\delta))$ .*

Corollary 1 should be compared to the case when  $C_1, \dots, C_k$  are chosen nonadaptively, for which the randomness complexity can be improved to  $n + O(\log k + \log(1/\delta))$  by applying the Goldreich-Wigderson randomness-efficient sampler for Boolean functions [GW97] and reusing randomness. The proof of Corollary 1 works by combining the GW sampler with our steward; the details are in Section 6.1.

$\varepsilon'$	$\delta'$	Randomness complexity	Reference
$\varepsilon$	$k\delta$	$nk$	Naïve
$O(\varepsilon d)$	$k\delta + \gamma$	$n + O(k \log(d+1) + (\log k) \log(1/\gamma))$	Theorem 1 (main)
$O(\varepsilon)$	$k\delta + \gamma$	$n + O(kd + (\log k) \log(1/\gamma))$	Theorem 4, $d_0 = 1$
$O(\varepsilon d)$	$k\delta + \gamma$	$n + k \log(d+2) + 2 \log(1/\gamma) + O(1)$	Theorem 3*
$O(\varepsilon d)$	$2^{O(k \log(d+1))} \cdot \delta$	$n$	Theorem 5, $d_0 = d$
$O(\varepsilon)$	$2^{O(kd)} \cdot \delta$	$n$	Theorem 5, $d_0 = 1$
$O(\varepsilon kd/\gamma)$	$k\delta + \gamma$	$n + O(k \log k + k \log d + k \log(1/\gamma))$	Prop. 1 (based on [SZ99])
Any	Any $\leq 0.2$	$n + \Omega(k) - \log(\delta'/\delta)$	Theorem 9 (lower bound)

\*Computationally inefficient.

Figure 2: Upper and lower bounds for stewards. Recall that  $\varepsilon, \delta$  are the concentration parameters of  $f_1, \dots, f_k$  (i.e. the error and failure probability of the estimation algorithm **Est**);  $\varepsilon', \delta'$  are the error and failure probability of the steward **S**;  $n$  is the number of input bits to each  $f_i$  (i.e. the number of random coins used by **Est**);  $k$  is the number of rounds of adaptivity;  $d$  is the dimension of the output of each  $f_i$  (i.e. the dimension of the output of **Est**.) Everywhere it appears,  $\gamma$  denotes an arbitrary positive number.

### 1.2.3 Application: Simulating an oracle for promise-BPP or APP

Recall that **promise-BPP** is the class of promise problems that can be decided in probabilistic polynomial time with bounded failure probability. When an algorithm is given oracle access to a promise problem, it is allowed to make queries that violate the promise, and several models have been considered for dealing with such queries. Following Moser [Mos01], we will stipulate that the oracle may respond in any arbitrary way to such queries. (See e.g. [BF99] for two other models.) From these definitions, it is easy to show, for example, that  $\text{BPP}^{\text{promise-BPP}} = \text{BPP}$ . Using our steward, we give a time- and randomness-efficient simulation of any algorithm with an oracle for **promise-BPP**. (As we will discuss in Section 1.4, the corresponding result for **BPP**-oracle algorithms is trivial.) The algorithm and analysis are almost identical to those used to prove Corollary 1. See Section 6.2 for details.

Recall that **APP**, introduced by Kabanets et al. [KRC00], is the class of functions  $\varphi : \{0, 1\}^n \rightarrow [0, 1]$  that can be approximated to within  $\pm \varepsilon$  in probabilistic  $\text{poly}(n, 1/\varepsilon)$  time with bounded failure probability. Following Moser [Mos01], we model oracle access to  $\varphi \in \text{APP}$  by requiring the oracle algorithm to provide  $w \in \{0, 1\}^n$  and a unary representation of  $1/\varepsilon \in \mathbb{N}$ ; the oracle is guaranteed to respond with a value that is within  $\pm \varepsilon$  of  $\varphi(w)$ . From these definitions, it is easy to show, for example, that  $\text{BPP}^{\text{APP}} = \text{BPP}$ . As with **promise-BPP**, we use our steward to construct a time- and randomness-efficient simulation of any algorithm with an oracle for **APP**. See Section 6.3 for details.

### 1.2.4 Application: The Goldreich-Levin algorithm

As a final application, in Section 6.4, we give a randomness-efficient version of the Goldreich-Levin algorithm [GL89] (otherwise known as the Kushilevitz-Mansour algorithm [KM93]) for finding noticeably large Fourier coefficients. Given oracle access to  $F : \{0, 1\}^n \rightarrow \{-1, 1\}$ , for any  $\theta > 0$ , we show how to efficiently find a list containing all  $U$  with  $|\hat{F}(U)| \geq \theta$ . Our algorithm makes  $O(n \log(n/\delta)) \cdot \text{poly}(1/\theta)$  queries to  $F$ , uses  $O(n + (\log n) \log(1/\delta))$  random bits, and has failure probability  $\delta$ . Notice that the number of random bits *does not depend on*  $\theta$ . To achieve such

a low randomness complexity, we first improve the randomness efficiency of each estimate in the standard Goldreich-Levin algorithm using the GW sampler. Then, we reduce the number of rounds of adaptivity by a factor of  $\log(1/\theta)$  by making many estimates within each round. Finally, we apply our steward with  $d = \text{poly}(1/\theta)$ , unlike our other applications where we choose  $d = 1$ . (Recall that  $d$  is the number of real values estimated in each round.)

### 1.2.5 Lower bound

We also give a *randomness complexity lower bound* of  $n + \Omega(k) - \log(\delta'/\delta)$  for any steward (Section 7). In the case  $d \leq O(1)$ , this comes close to matching our upper bounds. For example, to achieve  $\delta' \leq O(k\delta)$ , this lower bound says that  $n + \Omega(k)$  random bits are needed; our main steward (Theorem 1) achieves  $\varepsilon' \leq O(\varepsilon), \delta' \leq O(k\delta)$  using  $n + O(k + (\log k) \log(1/\delta))$  random bits. At the other extreme, if we want a steward that uses only  $n$  random bits, this lower bound says that the failure probability will be  $\delta' \geq \exp(\Omega(k)) \cdot \delta$ ; one of our variant stewards (Theorem 5) uses  $n$  random bits to achieve  $\varepsilon' \leq O(\varepsilon)$  and  $\delta' \leq \exp(O(k)) \cdot \delta$ .

## 1.3 Techniques

### 1.3.1 Block decision trees

A key component in the proof of our main result (Theorem 1) is a pseudorandom generator (PRG) for a new model that we call the *block decision tree* model. Informally, a block decision tree is a decision tree that reads its input from left to right,  $n$  bits at a time:

**Definition 2.** For a finite alphabet  $\Sigma$ , a  $(k, n, \Sigma)$  *block decision tree* is a rooted tree  $T = (V, E)$  of height  $k$  in which every node  $v$  at depth  $< k$  has exactly  $|\Sigma|$  children (labeled with the symbols in  $\Sigma$ ) and has an associated function  $v : \{0, 1\}^n \rightarrow \Sigma$ . We identify  $T$  with a function  $T : (\{0, 1\}^n)^{\leq k} \rightarrow V$  defined recursively:  $T(\text{the empty string}) = \text{the root node}$ , and if  $T(X_1, \dots, X_{i-1}) = v$ , then  $T(X_1, \dots, X_i)$  is the child of  $v$  labeled  $v(X_i)$ .

The standard nonconstructive argument (Appendix C) shows that there exists a  $\gamma$ -PRG for block decision trees with seed length  $n + k \log |\Sigma| + 2 \log(1/\gamma) + O(1)$ . (See Section 3.1 for the definition of a PRG in this setting.) In Section 3, we explicitly construct a  $\gamma$ -PRG for block decision trees with seed length  $n + O(k \log |\Sigma| + (\log k) \log(1/\gamma))$ . The generator is constructed by modifying the INW generator for space-bounded computation [INW94].

### 1.3.2 Shifting and rounding

For a steward  $S$ , let  $S(X)$  denote  $S$  using randomness  $X$ . Our main steward is of the form  $S(X) \stackrel{\text{def}}{=} S_0(\text{Gen}(X))$ . Here,  $\text{Gen}$  is our PRG for block decision trees, and  $S_0$  is a randomness-inefficient steward. In each round,  $S_0$  queries  $f_i$  at a fresh random point  $X_i \in \{0, 1\}^n$ , but  $S_0$  computes the return value  $Y_i$  by carefully *shifting and rounding* each coordinate of  $f_i(X_i)$ . In particular,  $S_0$  finds a single value  $\Delta_i$  such that after shifting each coordinate of  $f_i(X_i)$  according to  $\Delta_i$ , every coordinate is  $\varepsilon$ -far from every rounding boundary. Then,  $S_0$  rounds the shifted coordinates to obtain  $Y_i$ .

Roughly, the purpose of this shifting and rounding is to reduce the amount of information about  $X_i$  that is leaked by  $Y_i$ . To make this precise, observe that when any steward and owner interact, it is natural to model the owner's behavior by a decision tree that branches at each node based on the value  $Y_i$  provided by the steward. The *branching factor* of this decision tree is a simple measure of the amount of information leaked, and clearly, rounding  $f_i(X_i)$  reduces this branching factor. (Blum and Hardt [BH15] used a similar idea in a different setting.)

But more interestingly, we show that the branching factor can be reduced much further by *relaxing* the requirement that the tree perfectly computes  $O \leftrightarrow S_0$ . In particular, for every owner  $O$ , we construct a block decision tree  $T_O$  that merely *certifies correctness* of  $O \leftrightarrow S_0$ . That is, for any  $X_1, \dots, X_k$ , if the node  $T_O(X_1, \dots, X_k)$  indicates “success”, then the error  $\max_i \|Y_i - \mu_i\|_\infty$  in  $O \leftrightarrow S_0(X_1, \dots, X_k)$  is small. On the other hand, if  $T_O(X_1, \dots, X_k)$  does not indicate success, then “all bets are off”: the error  $\max_i \|Y_i - \mu_i\|_\infty$  in  $O \leftrightarrow S_0(X_1, \dots, X_k)$  may be small or large.

We show (Lemma 2) that our definition of  $S_0$  ensures the existence of a certification tree  $T_O$  with a branching factor of only  $d + 2$  with the additional property that

$$\Pr_{X_1, \dots, X_k} [T_O(X_1, \dots, X_k) \text{ indicates success}] \geq 1 - k\delta.$$

Therefore, to save random bits, we don’t need to try to fool  $O \leftrightarrow S_0$ . Instead, it suffices for **Gen** to fool  $T_O$ . The small branching factor of  $T_O$  allows **Gen** to have a correspondingly small seed length.

To construct the tree  $T_O$ , we think of  $\Delta_i$  as a *compressed* representation of  $Y_i$ . With high probability, given unlimited computation time,  $O$  could recover  $Y_i$  from  $\Delta_i$  by computing the *true* vector  $\mu_i$ , shifting it according to  $\Delta_i$ , and rounding. Each node of the certification tree  $T_O$ , therefore, just needs to have one child for each possible  $\Delta_i$  value, along with one  $\perp$  child indicating that the compression (and certification) failed.

### 1.3.3 Lower bound

Our lower bound follows a similar intuition as our upper bounds: we show that in each round, by carefully choosing  $f_i$ , the owner can learn  $\Omega(1)$  bits of information about the steward’s randomness. To conclude that the steward must use  $n + \Omega(k) - \log(\delta'/\delta)$  bits of randomness, we show that if the steward has fewer than  $n$  bits of randomness remaining from the owner’s perspective, then the owner can choose a function that causes the steward’s failure probability to be large.

## 1.4 Why can’t we just reuse the random bits?

Notwithstanding our lower bound, the reader might be tempted to think that randomness stewards are trivial: why not just pick  $X \in \{0, 1\}^n$  uniformly at random *once* and reuse it in every round? For the purpose of discussion, let us generalize, and suppose we are trying to execute an  $n$ -coin algorithm  $A$  (not necessarily an estimation algorithm) on  $k$  inputs  $C_1, \dots, C_k$ . If  $C_1, \dots, C_k$  are chosen *non-adaptively* (i.e. all in advance), then we really can use the same  $X$  for each execution. By the union bound, the probability that  $A(C_i, X)$  fails for any  $i$  is at most  $k\delta$ .

That argument breaks down in the adaptive case, because  $C_2$  is chosen based on  $A(C_1, X)$ , and hence  $C_2$  may be *stochastically dependent* on  $X$ , so  $A(C_2, X)$  is not guaranteed to have a low failure probability. Still, the argument can be salvaged in an important special case: Suppose  $A$  is a BPP algorithm. Then we can let  $\hat{C}_1, \hat{C}_2, \dots, \hat{C}_k$  be the inputs that would be chosen if  $A$  never failed. Then each  $\hat{C}_i$  really is independent of  $X$ , so by the union bound, with probability  $1 - k\delta$ ,  $A(\hat{C}_i, X)$  does not fail for any  $i$ . But if  $A(\hat{C}_i, X)$  does not fail for any  $i$ , then by induction,  $C_i = \hat{C}_i$  for every  $i$ . So the overall failure probability is once again at most  $k\delta$ .

The preceding argument applies more generally if  $A$  is *pseudodeterministic*, i.e. for each input, there is a unique correct output that  $A$  gives with probability  $1 - \delta$ .<sup>1</sup> (Pseudodeterministic algorithms were introduced by Gat and Goldwasser [GG11].) A BPP algorithm is trivially

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<sup>1</sup>These two conditions (inputs are chosen nonadaptively,  $A$  is pseudodeterministic) are both special cases of the following condition under which the randomness  $X$  may be safely reused: for every  $1 \leq i \leq k$ ,  $C_i$  is a pseudodeterministic function of  $(C_0, C_1, \dots, C_{i-1})$ , where  $C_0$  is a random variable that is independent of  $X$ .

pseudodeterministic. Observe, however, that a **promise-BPP** algorithm is only guaranteed to be pseudodeterministic on inputs that satisfy the promise. This is why the result we mentioned in Section 1.2.3 is in terms of an oracle for **promise-BPP**. Similarly, estimation algorithms (including APP algorithms) are typically not pseudodeterministic.

In the standard Goldreich-Levin algorithm, randomness is used to estimate  $\sum_{U \in \mathcal{U}} \hat{F}(U)^2$  for certain collections of subsets  $\mathcal{U}$ . The algorithm’s behavior depends on how the estimate compares to  $\theta^2/2$ . This process is not pseudodeterministic, because if the true value  $\sum_{U \in \mathcal{U}} \hat{F}(U)^2$  is very close to  $\theta^2/2$ , the estimate falls on each side of  $\theta^2/2$  with noticeable probability.

## 1.5 Related work

### 1.5.1 Adaptive data analysis

The notion of a randomness steward is inspired by the closely related *adaptive data analysis* problem [DFH<sup>+</sup>15c, BNS<sup>+</sup>15, DFH<sup>+</sup>15a, DFH<sup>+</sup>15b, HU14, SU14, BH15], introduced by Dwork et al. [DFH<sup>+</sup>15c]. In the simplest version of this problem, there is an unknown distribution  $\mathcal{D}$  over  $\{0, 1\}^n$  and a *data analyst* who wishes to estimate the mean values (with respect to  $\mathcal{D}$ ) of  $k$  adaptively chosen functions  $f_1, \dots, f_k : \{0, 1\}^n \rightarrow [0, 1]$  using as few samples from  $\mathcal{D}$  as possible. In this setting, these samples are held by a *mechanism* and not directly accessible by the data analyst. In round  $i$ , the data analyst gives  $f_i$  to the mechanism, and the mechanism responds with an estimate of  $\mathbb{E}_{x \sim \mathcal{D}}[f_i(x)]$ . The mechanism constructs the estimate so as to leak as little information as possible about the sample, so that the same sample points can be safely reused for future estimates.

The data analyst and mechanism in the adaptive data analysis setting are analogous to the owner  $\mathbf{O}$  and steward  $\mathbf{S}$  in our setting, respectively. In each case, the idea is that the mechanism or steward can intentionally introduce a small amount of error into each estimate to hide information and thereby facilitate future estimates. Note, however, that in the adaptive data analysis problem, there is just one unknown distribution  $\mathcal{D}$  and we are concerned with sample complexity, whereas in the randomness stewardship problem, we can think of each concentrated function  $f_i$  as defining a new distribution over  $\mathbb{R}^d$  and we are concerned with randomness complexity.

### 1.5.2 The Saks-Zhou algorithm

Another highly relevant construction is the algorithm of Saks and Zhou [SZ99] for simulating randomized logspace algorithms in deterministic space  $O(\log^{3/2} n)$ . The key component in this algorithm can be reinterpreted as a randomness steward. Using a pseudorandom generator, Saks and Zhou also constructed a randomized algorithm **Est** that approximates a large power of a given substochastic matrix. (Saks and Zhou used Nisan’s generator [Nis92], but any pseudorandom generator for small space can be used – see [Arm98, HU16].) By applying their steward, Saks and Zhou saved random bits when applying **Est** repeatedly to approximate a much larger power of a given substochastic matrix.

The “Saks-Zhou steward” works by *randomly perturbing and rounding* the output of each  $f_i$ , and then reusing the same randomness string  $X$  in each round. The perturbation and rounding are somewhat similar to our construction, but note that we shift the outputs of each  $f_i$  deterministically, whereas the Saks-Zhou steward uses random perturbations. The rounding parameters are also different. The analysis of the Saks-Zhou steward is similar to the proof that randomness can be safely reused for a pseudodeterministic subroutine; one can show that random perturbation and rounding effectively breaks the dependence between  $X$  and  $Y_i$ . (See Appendix B for the description and analysis of the Saks-Zhou steward.)

Our steward achieves better parameters than the Saks-Zhou steward (see Figure 2). In particular, to achieve failure probability  $k\delta + \gamma$ , the error  $\varepsilon'$  of the Saks-Zhou steward is  $O(\varepsilon kd/\gamma)$  – the error grows linearly with  $k$ , the number of rounds of adaptivity, as well as with  $1/\gamma$  – whereas our steward achieves error  $O(\varepsilon d)$ . Furthermore, the Saks-Zhou steward uses  $n + O(k \log k + k \log d + k \log(1/\gamma))$  random bits, whereas our steward uses only  $n + O(k \log(d+1) + (\log k) \log(1/\gamma))$  random bits.

### 1.5.3 Decision trees and branching programs

In the most common decision tree model, the branching factor  $|\Sigma|$  is just 2, and each node reads an arbitrary bit of the input. In the more general *parity decision tree* model, each node computes the parity of some subset of the input bits. Kushilevitz and Mansour showed [KM93] that the Fourier  $\ell_1$  norm of any Boolean function computed by a parity decision tree is at most  $2^k$ , the number of leaves in the tree. It is well-known (and easy to prove) that this implies that a  $\gamma$ -biased generator is a  $(2^k \gamma)$ -PRG for parity decision trees. Using e.g. the small-bias generator of Naor and Naor [NN93], this gives an efficient PRG for parity decision trees with asymptotically optimal seed length.

Decision trees in which each node computes a more complicated function have also been studied previously. Bellare [Bel92] introduced the *universal decision tree* model, in which each node computes an arbitrary Boolean function of the input bits. He gave a bound on the  $\ell_1$  norm of any Boolean function computed by a universal decision tree in terms of the  $\ell_1$  norms of the functions at each node. Unfortunately, for block decision trees, his bound is so large that it does not immediately imply any nontrivial pseudorandom generators for block decision trees.

A block decision tree can be thought of as a kind of space-bounded computation. Indeed, a block decision tree is a specific kind of *ordered branching program* of width  $|\Sigma|^k$  and length  $k$  that reads  $n$  bits at a time. Hence, we could directly apply a pseudorandom generator for ordered branching programs, such as the INW generator [INW94]. For these parameters, the INW generator has seed length of  $n + O(k \log k \log |\Sigma| + \log k \log(1/\gamma))$ . This seed length can be slightly improved by instead using Armoni’s generator [Arm98] (a generalization of the Nisan-Zuckerman generator [NZ96]), but even that slightly improved seed length is larger than the seed length of the generator we construct.

### 1.5.4 Finding noticeably large Fourier coefficients

Our randomness-efficient version of the Goldreich-Levin algorithm should be compared to the results of Bshouty et al. [BJT99], who gave several algorithms for finding noticeably large Fourier coefficients, all closely related to one another and based on an algorithm of Levin [Lev93].

- Bshouty et al. gave one algorithm [BJT99, Figure 4] that makes  $O(\frac{n}{\theta^2} \log(\frac{n}{\delta\theta}))$  queries and uses  $O(n \log(\frac{n}{\theta}) \log(\frac{1}{\delta\theta}))$  random bits. Our algorithm has better randomness complexity, but worse query complexity.
- Bshouty et al. gave another algorithm [BJT99, Figure 5] that makes only  $O(n/\theta^2)$  queries and uses just  $O(\log(n/\theta) \cdot \log(1/\theta))$  random bits, but it merely outputs a list such with probability  $1/2$ , some  $U$  in the list satisfies  $|\hat{F}(U)| \geq \theta$ , assuming such a  $U$  exists.

## 1.6 Outline of this paper

In Section 2, we describe the shifting and rounding steward  $S_0$  and prove that it admits certification trees with a small branching factor. Then, in Section 3, we construct and analyze our pseudorandom generator for block decision trees. In Section 4, we put these pieces together to prove our main result (Theorem 1). In Section 5, we show how to construct our variant stewards. In Section 6,



1. For  $i = 1$  to  $k$ :
  - (a)  $O$  chooses  $f_i : \{0, 1\}^n \rightarrow \mathbb{R}^d$  and gives it to  $M$ .
  - (b)  $S_0$  picks *fresh randomness*  $X_i \in \{0, 1\}^n$  and gives it to  $M$ .
  - (c)  $M$  gives  $W_i \stackrel{\text{def}}{=} f_i(X_i)$  to  $S_0$ .
  - (d)  $S_0$  computes  $Y_i$  by shifting and rounding  $W_i$  according to the algorithm in Section 2.1.
  - (e)  $S_0$  gives  $Y_i$  to  $O$ .

Figure 3: Outline of  $O \leftrightarrow S_0$ .

we explain our applications of our main steward. Finally, in Section 7, we prove our randomness complexity lower bound for stewards.

## 2 The shifting and rounding steward $S_0$

As a building block for our main steward constructions, we first construct our randomness-inefficient steward  $S_0$ . Recall that any steward makes two choices in each round: the input  $X_i$  to  $f_i$  and the estimate  $Y_i \in \mathbb{R}^d$ . The steward  $S_0$  focuses on the second choice: it queries each  $f_i$  at a fresh random point  $X_i \in \{0, 1\}^n$ , but it carefully shifts and rounds the output of  $f_i$ . (See Figure 3.)

### 2.1 The shifting and rounding algorithm

We now describe the algorithm by which  $S_0$  computes  $Y_i \in \mathbb{R}^d$  from  $W_i \stackrel{\text{def}}{=} f_i(X_i)$ . Fix  $n, k, d \in \mathbb{N}$  and  $\varepsilon, \delta > 0$ . Let  $[d]$  denote the set  $\{1, 2, \dots, d\}$ . Partition  $\mathbb{R}$  into half-open intervals of length  $(d+1) \cdot 2\varepsilon$ . For  $w \in \mathbb{R}$ , let  $\text{Round}(w)$  denote the midpoint of the interval containing  $w$ . Given  $W_i \in \mathbb{R}^d$ :

1. Find  $\Delta_i \in [d+1]$  such that for every  $j \in [d]$ , there is a single interval that entirely contains  $[W_{ij} + (2\Delta_i - 1)\varepsilon, W_{ij} + (2\Delta_i + 1)\varepsilon]$ . (We will show that such a  $\Delta_i$  exists.)
2. For every  $j \in [d]$ , set  $Y_{ij} = \text{Round}(W_{ij} + 2\Delta_i\varepsilon)$ .

We must show that this algorithm is well-defined:

**Lemma 1.** *For any  $W \in \mathbb{R}^d$ , there exists  $\Delta \in [d+1]$  such that for every  $j \in [d]$ , there is a single interval that entirely contains  $[W_j + (2\Delta - 1)\varepsilon, W_j + (2\Delta + 1)\varepsilon]$ .*

*Proof.* Consider picking  $\Delta \in [d+1]$  uniformly at random. Then for each  $j$ , the probability that  $[W_j + (2\Delta - 1)\varepsilon, W_j + (2\Delta + 1)\varepsilon]$  intersects two distinct intervals is precisely  $1/(d+1)$  by our choice of the length of the intervals. The union bound over  $d$  different  $j$  values completes the proof.  $\square$

### 2.2 Analysis: Certification trees

As outlined in Section 1.3.2, the key lemma says that for any owner  $O$ , there exists a block decision tree  $T_O$  with a small branching factor that certifies correctness of  $O \leftrightarrow S_0$ :

**Lemma 2.** Assume  $\delta < 1/2$ . Let  $\Sigma = [d+1] \cup \{\perp\}$ . For any deterministic owner  $\mathcal{O}$ , there exists a  $(k, n, \Sigma)$  block decision tree  $T_{\mathcal{O}}$  with the following properties.

1. For any internal node  $v$ ,  $\Pr_{X \in \{0,1\}^n}[v(X) = \perp] \leq \delta$ .
2. Fix  $X_1, \dots, X_k \in \{0,1\}^n$ , and suppose that the path from the root to  $T_{\mathcal{O}}(X_1, \dots, X_k)$  does not include any  $\perp$  nodes. Then  $\max_i \|Y_i - \mu_i\|_{\infty} \leq O(\varepsilon d)$  in  $\mathcal{O} \leftrightarrow S_0(X_1, \dots, X_k)$ .

From Lemma 2, it easily follows that if  $\text{Gen}$  is a  $\gamma$ -PRG for  $(k, n, \Sigma)$  block decision trees, then  $S_0(\text{Gen}(X))$  is an  $(O(\varepsilon d), k\delta + \gamma)$ -steward: the probability over  $X$  that  $T_{\mathcal{O}}(\text{Gen}(X))$  passes through any  $\perp$  nodes is at most  $k\delta + \gamma$ . Instantiating  $\text{Gen}$  with an explicit PRG with a short seed length will complete the proof of our main result. (See Section 4 for details.)

Notice that Lemma 2 does not assert that  $T_{\mathcal{O}}$  computes the transcript of  $\mathcal{O} \leftrightarrow S_0$ . In fact, we will construct  $T_{\mathcal{O}}$  to compute the transcript of a channel in a *different* protocol involving  $\mathcal{O}$  and  $S_0$ . In this new protocol, each  $Y_i$  chosen by  $S_0$  is *compressed and decompressed* before giving it to  $\mathcal{O}$ , as we suggested in Section 1.3.2. To facilitate the analysis, we introduce *two mediators*: one to compress and another to decompress.

*Proof of Lemma 2.*  $T_{\mathcal{O}}$  will be defined in terms of a thought experiment in which  $S_0$  interacts with  $\mathcal{O}$  through two mediators  $M_1$  and  $M_2$  as described above. The protocol is as follows:

1. For  $i = 1$  to  $k$ :
  - (a)  $\mathcal{O}$  chooses  $f_i : \{0,1\}^n \rightarrow \mathbb{R}^d$  that is  $(\varepsilon, \delta)$ -concentrated at some point  $\mu_i \in \mathbb{R}^d$  and gives it to  $M_1$ , who gives it to  $M_2$ .
  - (b)  $M_1$  and  $M_2$  both compute the smallest  $\hat{\mu}_i \in \mathbb{R}^d$  (under, say, the lexicographical order) such that  $f_i$  is  $(\varepsilon, \delta)$ -concentrated at  $\hat{\mu}_i$ . (This exists, because  $\{0,1\}^n$  is finite, so the set of points where  $f_i$  is concentrated is a closed subset of  $\mathbb{R}^d$ .)
  - (c)  $S_0$  chooses a string  $X_i \in \{0,1\}^n$  and gives it to  $M_2$ , who gives  $f_i(X_i) \in \mathbb{R}^d$  to  $S_0$ .
  - (d)  $S_0$  chooses  $Y_i \in \mathbb{R}^d$ . In the ordinary steward protocol,  $S_0$  gives  $Y_i$  to  $\mathcal{O}$ , but in the two-mediator protocol, the message is redirected to  $M_2$ .
  - (e) Say a value  $\Delta \in [d+1]$  is *compatible* with  $Y_i$  if  $Y_{ij} = \text{Round}(\hat{\mu}_{ij} + 2\Delta\varepsilon)$  for every  $j \in [d]$ .  $M_2$  computes

$$\hat{\Delta}_i = \begin{cases} \text{the smallest } \Delta \in [d+1] \text{ compatible with } Y_i & \text{if any such } \Delta \text{ exists} \\ \perp & \text{otherwise} \end{cases}$$

and gives it to  $M_1$ .

- (f)  $M_1$  computes  $\hat{Y}_i = (\hat{Y}_{i1}, \dots, \hat{Y}_{id})$ , where for each  $j \in [d]$ ,

$$\hat{Y}_{ij} = \begin{cases} \text{Round}(\hat{\mu}_{ij} + 2\hat{\Delta}_i\varepsilon) & \text{if } \hat{\Delta}_i \neq \perp \\ 0 & \text{otherwise} \end{cases}$$

and gives  $\hat{Y}_i$  to  $\mathcal{O}$ .

(See Figure 4.) Let  $\mathcal{O} \xleftrightarrow{2} S_0$  (“the two-mediator interaction of  $\mathcal{O}$  with  $S_0$ ”) denote the above interaction between  $\mathcal{O}$  and  $S_0$ .

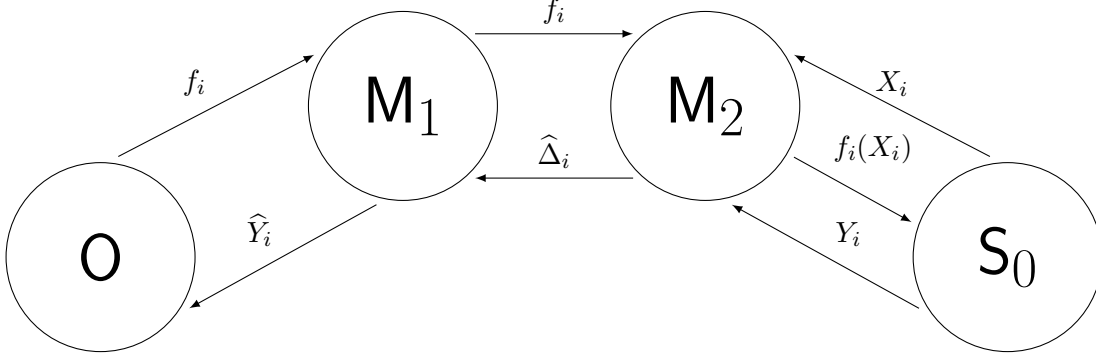


Figure 4: A single round of interaction in the thought experiment in the proof of Lemma 2.

**Definition of  $T_O$**  Let  $V$  be the vertex set of  $T_O$ . We first define  $T_O$  as a function. Because  $S_0(X_1, \dots, X_k)$  looks at  $X_i$  only in round  $i$ , we can sensibly speak of the first  $i$  rounds of  $O \xleftrightarrow{2} S_0(X_1, \dots, X_i)$  even for  $i < k$ . This allows us to define  $T_O(X_1, \dots, X_i)$  to be the node  $v \in V$  such that the path from the root to  $v$  is described by the values  $\hat{\Delta}_1, \dots, \hat{\Delta}_i$  that  $M_2$  sends in  $O \xleftrightarrow{2} S_0(X_1, \dots, X_i)$ .

Now, we must show that this function  $T_O$  can be realized as a block decision tree, i.e. that each internal node  $v \in V$  can be assigned a transition function  $v : \{0, 1\}^n \rightarrow \Sigma$  that is compatible with the definition of  $T_O$  as a function. Indeed, observe that  $\hat{\Delta}_1, \dots, \hat{\Delta}_{i-1}$  fully determine the state of  $O$  after the first  $i-1$  rounds of  $O \xleftrightarrow{2} S_0(X_1, \dots, X_i)$  and hence determines the function  $f_i$ . Furthermore,  $S_0$  is “memoryless”, i.e.  $Y_i$  is fully determined by  $f_i$  and  $X_i$ . Because of how  $M_2$  behaves, this shows that  $\hat{\Delta}_i$  is fully determined by  $\hat{\Delta}_1, \dots, \hat{\Delta}_{i-1}$  and  $X_i$ . So there is a function  $\varphi : (\hat{\Delta}_1, \dots, \hat{\Delta}_{i-1}, X_i) \mapsto \hat{\Delta}_i$ , and if the path from the root to  $v$  is described by  $\hat{\Delta}_1, \dots, \hat{\Delta}_{i-1}$ , we can set  $v(X_i) \stackrel{\text{def}}{=} \varphi(\hat{\Delta}_1, \dots, \hat{\Delta}_{i-1}, X_i)$ .

**Analysis of  $T_O$**  By the definition of  $T_O$  as a function, to prove Condition 1 in the lemma statement, we must show that in each round of  $O \xleftrightarrow{2} S_0$ ,  $\Pr[\hat{\Delta}_i = \perp] \leq \delta$ . Indeed, by concentration, with probability  $1 - \delta$ , for every  $j$ ,  $|W_{ij} - \hat{\mu}_{ij}| \leq \varepsilon$ . In this case, by the construction of  $S_0$ ,  $W_{ij} + 2\Delta_i\varepsilon$  and  $\hat{\mu}_{ij} + 2\Delta_i\varepsilon$  are in the same interval for every  $j \in [d]$ . Therefore, in this case, there is at least one value  $\Delta$  compatible with  $Y_i$ , namely the value  $\Delta_i$  used by  $S_0$ .

Finally, to prove Condition 2 in the lemma statement, suppose the path from the root node to  $T_O(X_1, \dots, X_k)$  does not include any  $\perp$  nodes. Then in  $O \xleftrightarrow{2} S_0(X_1, \dots, X_k)$ , for every  $i$ ,  $\hat{\Delta}_i \neq \perp$ . This implies that every  $Y_{ij}$  is of the form  $\text{Round}(\hat{\mu}_{ij} + 2\hat{\Delta}_i\varepsilon)$  for some  $\hat{\Delta}_i \in [d+1]$ . Therefore,  $|Y_{ij} - \hat{\mu}_{ij}| \leq 3(d+1)\varepsilon$ , since  $2\hat{\Delta}_i\varepsilon \leq 2(d+1)\varepsilon$  and rounding introduces at most  $(d+1)\varepsilon$  additional error. Since  $\delta < 1/2$ ,  $\|\hat{\mu}_i - \mu_i\|_\infty \leq 2\varepsilon$ , so  $\|Y_i - \mu_i\|_\infty \leq 3(d+1)\varepsilon + 2\varepsilon = (3d+5)\varepsilon$ .

Of course, that is still all in  $O \xleftrightarrow{2} S_0(X_1, \dots, X_k)$ . But the crucial point is, for every  $i$ , since  $\hat{\Delta}_i \neq \perp$ , we can be sure that  $Y_i = \hat{Y}_i$ . Therefore, the values  $\mu_1, \dots, \mu_k, Y_1, \dots, Y_k$  in  $O \xleftrightarrow{2} S_0(X_1, \dots, X_k)$  are *exactly the same* as they are in  $O \leftrightarrow S_0(X_1, \dots, X_k)$ ! Therefore, in  $O \leftrightarrow S_0(X_1, \dots, X_k)$ , for every  $i$ ,  $\|Y_i - \mu_i\|_\infty \leq (3d+5)\varepsilon$ .  $\square$

Notice that in  $O \xleftrightarrow{2} S_0(X_1, \dots, X_k)$ , if  $\hat{\Delta}_i = \perp$  for some  $i$ , then the interaction might diverge from  $O \leftrightarrow S_0(X_1, \dots, X_k)$ , in which case  $T_O(X_1, \dots, X_k)$  does not encode the transcript of  $O \leftrightarrow S_0(X_1, \dots, X_k)$  in any way.

### 3 Pseudorandomness for block decision trees

Recall that our goal is to modify the internal parameters of the INW generator, thereby constructing a  $\gamma$ -PRG for  $(k, n, \Sigma)$  block decision trees with seed length  $n + O(k \log |\Sigma| + (\log k) \log(1/\gamma))$ . The construction and analysis mimic the standard treatment of the INW generator, and the reader who is familiar with the INW generator is encouraged to skip to Section 3.4 to just see the new parameters.

#### 3.1 Formal definitions and theorem statement

Let  $U_n$  denote the uniform distribution on  $\{0, 1\}^n$ . For two probability distributions  $\mu, \mu'$  on the same measurable space, write  $\mu \sim_\gamma \mu'$  to indicate that  $\mu$  and  $\mu'$  have total variation distance at most  $\gamma$ .

**Definition 3.** We say that  $\text{Gen} : \{0, 1\}^s \rightarrow \{0, 1\}^{nk}$  is a  $\gamma$ -PRG for  $(k, n, \Sigma)$  block decision trees if for every such tree  $T$ ,  $T(\text{Gen}(U_s)) \sim_\gamma T(U_{nk})$ .

**Theorem 2.** For every  $n, k \in \mathbb{N}$ , every finite alphabet  $\Sigma$ , and every  $\gamma > 0$ , there exists a  $\gamma$ -PRG  $\text{Gen} : \{0, 1\}^s \rightarrow \{0, 1\}^{nk}$  for  $(k, n, \Sigma)$  block decision trees with seed length

$$s \leq n + O(k \log |\Sigma| + (\log k) \log(1/\gamma)).$$

The PRG can be computed in  $\text{poly}(n, k, \log |\Sigma|, \log(1/\gamma))$  time.

#### 3.2 Concatenating PRGs for block decision trees

Toward proving Theorem 2, for a  $(k, n, \Sigma)$  block decision tree  $T = (V, E)$  and a node  $v \in V$ , let  $T_v$  denote the subtree rooted at  $v$ , and observe that we can think of  $T_v$  as a  $(k', n, \Sigma)$  block decision tree, where  $k' = k - \text{depth}(v)$ . This simple observation – after a block decision tree has been computing for a while, the remaining computation is just another block decision tree – implies that pseudorandom generators for block decision trees can be *concatenated* with mild error accumulation. This fact and its easy proof are perfectly analogous to the situation with ordered branching programs. We record the details below.

**Lemma 3.** Suppose  $\text{Gen}_1 : \{0, 1\}^{s_1} \rightarrow \{0, 1\}^{nk_1}$  is a  $\gamma_1$ -PRG for  $(k_1, n, \Sigma)$  block decision trees and  $\text{Gen}_2 : \{0, 1\}^{s_2} \rightarrow \{0, 1\}^{nk_2}$  is a  $\gamma_2$ -PRG for  $(k_2, n, \Sigma)$  block decision trees. Let  $\text{Gen}(x, y) = (\text{Gen}_1(x), \text{Gen}_2(y))$ . Then  $\text{Gen}$  is a  $(\gamma_1 + \gamma_2)$ -PRG for  $(k_1 + k_2, n, \Sigma)$  block decision trees.

*Proof.* Fix a  $(k_1 + k_2, n, \Sigma)$  block decision tree  $T$ . For a node  $u$  at depth  $k_1$  and a leaf node  $v$ , define

$$\begin{aligned} p(u) &= \Pr[T(U_{nk_1}) = u] & p(v | u) &= \Pr[T_u(U_{nk_2}) = v] \\ \tilde{p}(u) &= \Pr[T(\text{Gen}_1(U_{s_1})) = u] & \tilde{p}(v | u) &= \Pr[T_u(\text{Gen}_2(U_{s_2})) = v]. \end{aligned}$$

To prove correctness of  $\text{Gen}$ , recall that  $\ell_1$  distance is twice total variation distance. The  $\ell_1$  distance between  $T(\text{Gen}(U_{s_1+s_2}))$  and  $T(U_{n(k_1+k_2)})$  is precisely  $\sum_{u,v} |p(u)p(v | u) - \tilde{p}(u)\tilde{p}(v | u)|$ . By the triangle inequality, this is bounded by

$$\begin{aligned} & \sum_{u,v} |p(u)p(v | u) - p(u)\tilde{p}(v | u)| + \sum_{u,v} |p(u)\tilde{p}(v | u) - \tilde{p}(u)\tilde{p}(v | u)| \\ &= \sum_{u,v} p(u) \cdot |p(v | u) - \tilde{p}(v | u)| + \sum_{u,v} |p(u) - \tilde{p}(u)| \cdot \tilde{p}(v | u) \\ &= \sum_u p(u) \sum_v |p(v | u) - \tilde{p}(v | u)| + \sum_u |p(u) - \tilde{p}(u)|. \end{aligned}$$

By the correctness of  $\text{Gen}_1$  and  $\text{Gen}_2$ , this is bounded by  $(\sum_u p(u) \cdot 2\gamma_2) + 2\gamma_1 = 2(\gamma_1 + \gamma_2)$ .  $\square$

### 3.3 Recycling randomness

We find it most enlightening to think of the INW generator in terms of extractors, as suggested by Raz and Reingold [RR99] and in the spirit of the Nisan-Zuckerman generator [NZ96]. The analysis is particularly clean if we work with *average-case extractors*, a concept introduced by Dodis et al. [DORS08].

**Definition 4.** For discrete random variables  $X, V$ , the *average-case conditional min-entropy* of  $X$  given  $V$  is

$$\tilde{H}_\infty(X | V) = -\log_2 \left( \mathbb{E}_{v \sim V} \left[ 2^{-H_\infty(X|V=v)} \right] \right),$$

where  $H_\infty$  is (standard) min-entropy.

Intuitively,  $\tilde{H}_\infty(X | V)$  measures the amount of randomness in  $X$  from the perspective of someone who knows  $V$ . The output of an *average-case extractor* is required to look uniform even from the perspective of someone who knows  $V$ , as long as its first input is sampled from a distribution that has high min-entropy conditioned on  $V$ :

**Definition 5.** We say that  $\text{Ext} : \{0, 1\}^s \times \{0, 1\}^d \rightarrow \{0, 1\}^m$  is an *average-case  $(s - t, \beta)$ -extractor* if for every  $X$  distributed on  $\{0, 1\}^s$  and every discrete random variable  $V$  such that  $\tilde{H}_\infty(X | V) \geq s - t$ , if we let  $Y \sim U_d$  be independent of  $(X, V)$  and let  $Z \sim U_m$  be independent of  $V$ , then  $(V, \text{Ext}(X, Y)) \sim_\beta (V, Z)$ .

Average-case extractors are the perfect tools for *recycling randomness* in space-bounded computation. We record the details for block decision trees below.

**Lemma 4** (Randomness recycling lemma for block decision trees). *Suppose  $\text{Gen} : \{0, 1\}^s \rightarrow \{0, 1\}^{nk}$  is a  $\gamma$ -PRG for  $(k, n, \Sigma)$  block decision trees and  $\text{Ext} : \{0, 1\}^s \times \{0, 1\}^d \rightarrow \{0, 1\}^s$  is an average-case  $(s - k \log |\Sigma|, \beta)$ -extractor. Define*

$$\text{Gen}'(x, y) = (\text{Gen}(x), \text{Gen}(\text{Ext}(x, y))).$$

*Then  $\text{Gen}'$  is a  $(2\gamma + \beta)$ -PRG for  $(2k, n, \Sigma)$  block decision trees.*

*Proof.* Let  $T$  be a  $(2k, n, \Sigma)$  block decision tree. Let  $X \sim U_s$  and let  $V = T(\text{Gen}(X))$ . By [DORS08, Lemma 2.2b], the fact that  $V$  can be described using  $k \log |\Sigma|$  bits implies that  $\tilde{H}_\infty(X | V) \geq s - k \log |\Sigma|$ . Therefore, by the average-case extractor condition, if we let  $Y \sim U_d$  be independent of  $X$  and  $Z \sim U_d$  be independent of  $V$ , then

$$(V, \text{Ext}(X, Y)) \sim_\beta (V, Z).$$

Applying a (deterministic) function can only make the distributions closer. Apply the function  $(v, z) \mapsto T_v(\text{Gen}(z))$ :

$$T(\text{Gen}'(X, Y)) \sim_\beta T(\text{Gen}(X), \text{Gen}(Z)).$$

By Lemma 3, the right-hand side is  $(2\gamma)$ -close to  $T(U_{2nk})$ . The triangle inequality completes the proof.  $\square$

To actually construct a generator, we will need to instantiate this randomness recycling lemma with an explicit average-case extractor:

**Lemma 5.** *For every  $s, t \in \mathbb{N}$  and every  $\beta > 0$ , there exists an average-case  $(s - t, \beta)$ -extractor  $\text{Ext} : \{0, 1\}^s \times \{0, 1\}^d \rightarrow \{0, 1\}^s$  with seed length  $d \leq O(t + \log(1/\beta))$  computable in time  $\text{poly}(s, \log(1/\beta))$ .*

*Proof sketch.* It is standard (and can be proven using expanders, see e.g. [Vad12]) that there exists an *ordinary*  $(s - t - \log(2/\beta), \beta/2)$ -extractor  $\text{Ext} : \{0, 1\}^s \times \{0, 1\}^d \rightarrow \{0, 1\}^s$  with seed length  $d \leq O(t + \log(1/\beta))$  computable in time  $\text{poly}(s, \log(1/\beta))$ . By the same argument as that used to prove [DORS08, Lemma 2.3],  $\text{Ext}$  is automatically an average-case  $(s - t, \beta)$ -extractor.  $\square$

### 3.4 The recursive construction

*Proof of Theorem 2.* Define  $\beta = \gamma/2^{\lceil \log k \rceil}$ . For  $i \geq 0$ , define  $s_i \in \mathbb{N}$ ,  $d_i \in \mathbb{N}$ ,  $G_i : \{0, 1\}^{s_i} \rightarrow \{0, 1\}^{n \cdot 2^i}$ , and  $\text{Ext}_i : \{0, 1\}^{s_i} \times \{0, 1\}^{d_i} \rightarrow \{0, 1\}^{s_i}$  through mutual recursion as follows. Start with  $s_0 = n$  and  $G_0(x) = x$ . Having already defined  $s_i$  and  $G_i$ , let  $\text{Ext}_i$  be the average-case  $(s_i - 2^i \log |\Sigma|, \beta)$ -extractor of Lemma 5, and let  $d_i$  be its seed length. Then let  $s_{i+1} = s_i + d_i$ , and let

$$G_{i+1}(x, y) = (G_i(x), G_i(\text{Ext}_i(x, y))).$$

We show by induction on  $i$  that  $G_i$  is a  $(\beta \cdot (2^i - 1))$ -PRG for  $(2^i, n, \Sigma)$  block decision trees. In the base case  $i = 0$ , this is trivial. For the inductive step, apply Lemma 4, and note that  $2\beta(2^i - 1) + \beta = \beta(2^{i+1} - 1)$ . This completes the induction. Therefore, we can let  $\text{Gen} = G_{\lceil \log k \rceil}$ , since  $\beta \cdot (2^{\lceil \log k \rceil} - 1) < \gamma$ . The seed length  $s_{\lceil \log k \rceil}$  of  $\text{Gen}$  is

$$\begin{aligned} n + \sum_{i=0}^{\lceil \log k \rceil} d_i &\leq n + O \left( \sum_{i=0}^{\lceil \log k \rceil} (2^i \log |\Sigma| + \log k + \log(1/\gamma)) \right) \\ &\leq n + O(k \log |\Sigma| + (\log k) \log(1/\gamma)). \end{aligned}$$

The time needed to compute  $\text{Gen}(x)$  is just the time needed for  $O(k)$  applications of  $\text{Ext}_i$  for various  $i \leq O(\log k)$ , which is  $\text{poly}(n, k, \log |\Sigma|, \log(1/\gamma))$ .  $\square$

## 4 Proof of main result

Without loss of generality, assume  $\delta < 1/2$ . (If  $\delta \geq 1/2$ , then either  $k = 1$  or  $k\delta \geq 1$ ; in either case, the result is trivial.) Let  $S_0$  be the steward of Section 2, let  $\Sigma$  be the alphabet of Lemma 2, and let  $\text{Gen}$  be the  $\gamma$ -PRG for  $(k, n, \Sigma)$  block decision trees of Theorem 2. The steward is  $S(X) \stackrel{\text{def}}{=} S_0(\text{Gen}(X))$ .

Consider any owner  $O$ . We may assume without loss of generality that  $O$  is deterministic, because a randomized owner is just a distribution over deterministic owners. By Condition 2 of Lemma 2 and the union bound,

$$\Pr[\text{some node in the path from the root to } T_O(U_{nk}) \text{ is labeled } \perp] \leq k\delta.$$

Therefore, when  $T_O$  reads  $\text{Gen}(U_s)$  instead of  $U_{nk}$ , the probability is at most  $k\delta + \gamma$ . By Condition 2 of Lemma 2, this proves the correctness of  $S$ . The randomness complexity of  $S$  is just the seed length of  $\text{Gen}$ , which is indeed  $n + O(k \log |\Sigma| + (\log k) \log(1/\gamma)) = n + O(k \log(d + 1) + (\log k) \log(1/\gamma))$ . The total runtime of  $S$  is clearly  $\text{poly}(n, k, d, \log(1/\varepsilon), \log(1/\gamma))$ .  $\square$

## 5 Variant stewards

**Theorem 3.** *For any  $n, k, d \in \mathbb{N}$ , for any  $\varepsilon, \delta, \gamma > 0$ , there exists a (computationally inefficient)  $(O(\varepsilon d), k\delta + \gamma)$ -steward for  $k$  adaptively chosen  $(\varepsilon, \delta)$ -concentrated functions  $f_1, \dots, f_k : \{0, 1\}^n \rightarrow \mathbb{R}^d$  with randomness complexity*

$$n + k \log(d + 2) + 2 \log(1/\gamma) + O(1).$$

*Proof sketch.* Mimic the proof of Theorem 1, but use a PRG obtained by the standard nonconstructive argument (Appendix C).  $\square$

The shifting and rounding steward  $S_0$  can be generalized to achieve a tradeoff between low error  $\varepsilon'$  and low branching factor  $|\Sigma|$  of the certification tree  $T_O$ . In particular, for any factorization  $d = d_0 d_1$ , one can reduce the error from  $O(\varepsilon d)$  down to  $O(\varepsilon d_0)$  at the cost of increasing the branching factor of  $T_O$  from  $d + 2$  up to  $(d_0 + 1)^{d_1} + 1$ . This is achieved by simply partitioning the  $d$  coordinates into  $d_1$  groups of  $d_0$  coordinates and shifting each group individually; the details are in Appendix A. This immediately implies the following generalization of Theorem 1, which achieves a tradeoff between error and randomness complexity:

**Theorem 4.** *For any  $n, k, d, d_0 \in \mathbb{N}$  with  $d_0 \leq d$ , for any  $\varepsilon, \delta, \gamma > 0$ , there exists an  $(O(\varepsilon d_0), k\delta + \gamma)$ -steward for  $k$  adaptively chosen  $(\varepsilon, \delta)$ -concentrated functions  $f_1, \dots, f_k : \{0, 1\}^n \rightarrow \mathbb{R}^d$  with randomness complexity*

$$n + O\left(\frac{kd \log(d_0 + 1)}{d_0} + (\log k) \log(1/\gamma)\right).$$

*The total running time of the steward is  $\text{poly}(n, k, d, \log(1/\varepsilon), \log(1/\gamma))$ .*

Recall from the introduction that if  $f_1, \dots, f_k$  are chosen nonadaptively, then we can reuse randomness and just union bound over the  $k$  functions. We now show that we can reuse the randomness in  $S_0$ , as long as we union bound over *all the nodes* in the certification tree. (This is similar to the analysis of the Saks-Zhou steward, except that in the Saks-Zhou case, the branching factor of the tree is just 1. It is also similar to the analysis in [BH15].) This gives a steward with very low randomness complexity but large failure probability:

**Theorem 5.** *For any  $n, k, d, d_0 \in \mathbb{N}$  with  $d_0 \leq d$ , for any  $\varepsilon, \delta > 0$ , there exists an  $(O(\varepsilon d_0), \delta')$ -steward for  $k$  adaptively chosen  $(\varepsilon, \delta)$ -concentrated functions  $f_1, \dots, f_k : \{0, 1\}^n \rightarrow \mathbb{R}^d$  with randomness complexity  $n$ , where*

$$\delta' \leq \exp\left(O\left(\frac{kd \log(d_0 + 1)}{d_0}\right)\right) \cdot \delta.$$

*The total running time of the steward is  $\text{poly}(n, k, d, \log(1/\varepsilon))$ .*

*Proof.* Assume without loss of generality that  $d$  is a multiple of  $d_0$  and that  $\delta < 1/2$ . The steward is  $S(X) \stackrel{\text{def}}{=} S_0(X, X, X, \dots, X)$ , where  $S_0$  is the steward of Section 2 generalized as in Appendix A. To prove correctness, fix any deterministic owner  $O$ . Let  $T_O$  be the block decision tree of Lemma 11. By Condition 1 of Lemma 11, from any internal node, if  $T_O$  reads  $X$ , the probability that it moves to the  $\perp$  child is at most  $\delta$ . Therefore, by the union bound over all nodes, the probability that there is some node from which  $T_O$  would move to the  $\perp$  child upon reading  $X$  is at most the value  $\delta'$  in the lemma statement. By Condition 2 of Lemma 11, if no node in  $T_O$  takes a  $\perp$  transition upon reading  $X$ , then  $\max_i \|\mu_i - Y_i\|_\infty \leq O(\varepsilon d_0)$  in  $O \leftrightarrow S(X)$ .  $\square$

## 6 Applications

### 6.1 Acceptance probabilities of Boolean circuits

A  $(\varepsilon, \delta)$ -sampler for Boolean functions on  $n$  bits is a randomized oracle algorithm **Samp** such that for any Boolean function  $C : \{0, 1\}^n \rightarrow \{0, 1\}$ , if we let  $\mu(C) \stackrel{\text{def}}{=} 2^{-n} \sum_x C(x)$  be the acceptance probability of  $C$ , then  $\Pr[|\mathbf{Samp}^C - \mu(C)| > \varepsilon] \leq \delta$ . We will use a near-optimal sampler constructed by Goldreich and Wigderson [GW97]:

**Lemma 6** ([GW97, Theorem 6.5]). *For every  $n \in \mathbb{N}$  and every  $\varepsilon, \delta > 0$ , there is an  $(\varepsilon, \delta)$ -sampler for Boolean functions on  $n$  bits that makes  $O(\log(1/\delta)/\varepsilon^2)$  queries, uses  $n + O(\log(1/\delta))$  random bits, and runs in time  $\text{poly}(n, 1/\varepsilon, \log(1/\delta))$ .*

*Proof of Corollary 1.* Let  $c$  be the constant under the  $O(\cdot)$  of the error  $\varepsilon'$  in the steward of Theorem 1. When given parameters  $n, k, \varepsilon, \delta$ , let **Samp** be the Boolean  $(\varepsilon/c, \delta/(2k))$ -sampler of Lemma 6, and say it uses  $m$  coins. Let **S** be the  $(\varepsilon, \delta)$ -steward of Theorem 1 for  $k$  adaptively chosen  $(\varepsilon/c, \delta/(2k))$ -concentrated functions  $f_1, \dots, f_k : \{0, 1\}^m \rightarrow \mathbb{R}$ . (So  $\gamma = \delta/2$ .) When given circuit  $C_i$ , define  $f_i(X) = \mathbf{Samp}^{C_i}(X)$ , i.e. the output  $\mathbf{Samp}^{C_i}$  with randomness  $X$ . Give  $f_i$  to **S**, and output the value  $Y_i$  that it returns.

*Proof of correctness:* The definition of a sampler implies that each  $f_i$  is  $(\varepsilon/c, \delta/(2k))$ -concentrated at  $\mu(C_i)$ . Furthermore, each  $f_i$  is defined purely in terms of  $C_i$ , which is chosen based only on  $Y_1, \dots, Y_{i-1}$ . Therefore, the steward guarantee implies that with probability  $1 - \delta$ , every  $Y_i$  is within  $\pm \varepsilon$  of  $\mu(C_i)$ .

*Randomness complexity analysis:* The number of bits  $m$  used by the sampler is  $n + O(\log(k/\delta))$ . Therefore, the number of bits used by the steward is

$$n + O(\log(k/\delta)) + O(k + (\log k) \log(1/\delta)) = n + O(k + (\log k) \log(1/\delta)).$$

*Runtime analysis:* The runtime of the steward is  $\text{poly}(m, k, \log(1/\gamma)) = \text{poly}(n, k, \log(1/\delta))$ . The runtime of the sampler is  $\text{poly}(n, 1/\varepsilon, \log k, \log(1/\delta))$ . The time required to evaluate each query of the sampler in round  $i$  is  $O(\text{size}(C_i))$  (assuming we work with a suitable computational model and a suitable encoding of Boolean circuits.) The number of queries that the sampler makes in each round is  $O(\log(k/\delta)/\varepsilon^2)$ . Therefore, the total runtime of this algorithm is

$$O\left(\frac{\log k + \log(1/\delta)}{\varepsilon^2} \cdot \sum_{i=1}^k \text{size}(C_i)\right) + \text{poly}(n, k, 1/\varepsilon, \log(1/\delta)). \quad \square$$

### 6.2 Simulating a promise-BPP oracle

**Theorem 6.** *Suppose a search problem  $\Pi$  can be solved by a deterministic promise-BPP-oracle algorithm that runs in time  $T$  and makes  $k$  queries, and suppose that (regardless of previous oracle responses) each query of this algorithm can be decided by a randomized algorithm that runs in time  $T'$ , uses  $n$  coins, and has failure probability  $1/3$ . Then for any  $\delta$ ,  $\Pi$  can be solved by a randomized (non-oracle) algorithm that runs in time*

$$T + O(T' \cdot k \log(k/\delta)) + \text{poly}(n, k, \log(1/\delta)),$$

*has randomness complexity*

$$n + O(k + (\log k) \log(1/\delta)),$$

*and has failure probability  $\delta$ .*



(Recall that search problems generalize decision problems and function problems. In reality, the theorem generalizes to just about any kind of “problem”, but we restrict ourselves to search problems for concreteness.) The theorem can easily be extended to randomized oracle algorithms by considering the problem of executing the randomized oracle algorithm using a given randomness string.

As a reminder, as discussed in Section 1.4, Theorem 6 would be trivial if it involved a BPP oracle instead of a **promise-BPP** oracle. Indeed, in the BPP case, the randomness can be reduced to just  $n + O(\log k + \log(1/\delta))$ . This is because a BPP algorithm is pseudodeterministic, so the randomness can be safely reused from one query to the next. A **promise-BPP** algorithm is not pseudodeterministic in general – it is only guaranteed to be pseudodeterministic on inputs that satisfy the promise.

*Proof sketch of Theorem 6.* Let  $B$  be the algorithm of Corollary 1 with  $\varepsilon = 1/10$  and the desired failure probability  $\delta$ . When the oracle algorithm makes query  $i$ , define  $f_i(X)$  to be the value outputted by the **promise-BPP** algorithm on that query string using randomness  $X$ . Give  $B$  the “circuit”  $f_i$ . (The algorithm  $B$  treats the circuits as black boxes, so we don’t need to bother implementing  $f_i$  as a literal Boolean circuit; the important thing is that  $f_i(X)$  can be evaluated in time  $T'$ .) When  $B$  outputs a value  $Y_i$ , give the oracle algorithm the response 0 if  $Y_i < 1/2$  and 1 if  $Y_i \geq 1/2$ .  $\square$

### 6.3 Simulating an APP oracle

**Theorem 7.** *Suppose  $\varphi \in \text{APP}$  and a search problem  $\Pi$  can be solved by a deterministic  $\varphi$ -oracle algorithm that runs in time  $T$  and makes  $k$  queries  $(w_1, \varepsilon), \dots, (w_k, \varepsilon)$  (where  $w_i$  depends on previous oracle responses, but  $\varepsilon$  is the same for every query.) Let  $c$  be the constant under the  $O(\cdot)$  in the error  $\varepsilon'$  in Theorem 1. Suppose that (regardless of the oracle responses)  $\varphi(w_i)$  can be approximated to within  $\pm \varepsilon/c$  by a randomized algorithm that runs in time  $T'$ , uses  $n$  coins, and has failure probability  $1/3$ . Then for any  $\delta$ ,  $\Pi$  can be solved by a randomized (non-oracle) algorithm that runs in time*

$$T + O(T' \cdot k \log(k/\delta)) + \text{poly}(n, k, \log(1/\delta)),$$

*has randomness complexity*

$$n + O(k + (\log k) \log(1/\delta)),$$

*and has failure probability  $\delta$ .*

The proof of Theorem 7 is similar to the proofs of Corollary 1 and Theorem 6. The difference is that a sampler as defined previously is no longer quite the right tool for deterministic amplification; to amplify an APP algorithm, we are not trying to estimate the *acceptance probability* of a Boolean function, but rather the point where a  $[0, 1]$ -valued function is *concentrated*. For this, we use an *averaging sampler*.

An *averaging  $(\varepsilon, \delta)$ -sampler* for Boolean functions on  $n$  bits is an algorithm  $\text{Samp} : \{0, 1\}^m \rightarrow \{0, 1\}^t$  such that for any Boolean function  $C : \{0, 1\}^n \rightarrow \{0, 1\}$ , if we let  $\mu(C) \stackrel{\text{def}}{=} 2^{-n} \sum_x C(x)$  be the acceptance probability of  $C$ , then

$$\Pr_{X \in \{0, 1\}^m} \left[ \left| \mu(C) - \frac{1}{t} \sum_{i=1}^t C(\text{Samp}(X)_i) \right| > \varepsilon \right] \leq \delta.$$

(Note that an averaging sampler induces a sampler of a very specific form: query the oracle at several points and output the empirical mean.) We now show that an averaging sampler can be used to

decrease the failure probability of a concentrated function by taking a *median*. This observation (in a different form) is due to Bellare, Goldreich, and Goldwasser [BGG93].

**Lemma 7.** *Suppose  $f : \{0, 1\}^n \rightarrow \mathbb{R}$  is  $(\varepsilon, \delta_0)$ -concentrated at  $\mu \in \mathbb{R}$  and  $\text{Samp} : \{0, 1\}^m \rightarrow (\{0, 1\}^n)^t$  is an averaging  $(\varepsilon', \delta)$ -sampler for Boolean functions on  $n$  bits, where  $\varepsilon' + \delta_0 < 1/2$ . Define  $g : \{0, 1\}^m \rightarrow \mathbb{R}$  by*

$$g(x) = \text{median}_{i \in [t]} f(\text{Samp}(x)_i).$$

*Then  $g$  is  $(\varepsilon, \delta)$ -concentrated at  $\mu$ .*

*Proof.* Let  $C : \{0, 1\}^n \rightarrow \{0, 1\}$  be the indicator function for  $\{x : |f(x) - \mu| \leq \varepsilon\}$ . Then by the concentration of  $f$ ,  $2^{-n} \sum_x C(x) \geq 1 - \delta_0$ . Therefore, by the averaging sampler condition, with probability  $1 - \delta$  over  $x$ ,  $\frac{1}{t} \sum_i C(\text{Samp}(X)_i) \geq 1 - \delta_0 - \varepsilon' > 1/2$ . If this is the case, then more than half of the values  $f(\text{Samp}(x)_1), \dots, f(\text{Samp}(x)_t)$  are within  $\pm \varepsilon$  of  $\mu$ , which implies that their median is within  $\pm \varepsilon$  of  $\mu$ .  $\square$

The following lemma gives the parameters achieved by the famous “random walk on expanders” averaging sampler; see e.g. [Vad12, Corollary 4.41].

**Lemma 8.** *For every  $n \in \mathbb{N}$  and every  $\varepsilon, \delta > 0$ , there is an averaging  $(\varepsilon, \delta)$ -sampler for Boolean functions on  $n$  bits with  $m \leq n + O(\log(1/\delta)/\varepsilon^2)$  and  $t \leq O(\log(1/\delta)/\varepsilon^2)$ , computable in time  $\text{poly}(n, 1/\varepsilon, \log(1/\delta))$ .*

**Corollary 2** (Deterministic amplification for APP). *Suppose  $\varphi \in \text{APP}$  via an algorithm that on input  $(w, \varepsilon)$  uses  $n$  coins and  $t$  time steps to compute  $\varphi(w) \pm \varepsilon$  with failure probability  $1/3$ . Then for any  $\delta$ , is possible to compute  $\varphi(w) \pm \varepsilon$  with failure probability  $\delta$  using  $O(t \log(1/\delta)) + \text{poly}(n, \log(1/\delta))$  time steps and  $n + O(\log(1/\delta))$  coins.*

*Proof.* On input  $(w, \varepsilon)$ :

1. Let  $\text{Samp} : \{0, 1\}^m \rightarrow (\{0, 1\}^n)^t$  be the averaging  $(1/10, \delta)$ -sampler for Boolean functions on  $n$  bits of Lemma 8.
2. Define  $f : \{0, 1\}^n \rightarrow [0, 1]$  by letting  $f(X)$  be the output of the  $1/3$ -error-probability algorithm for computing  $\varphi(w) \pm \varepsilon$  on randomness  $X$ .
3. Pick  $X \in \{0, 1\}^m$  uniformly at random and return  $\text{median}_{i \in [t]} f(\text{Samp}(X)_i)$ .

Correctness follows immediately from Lemma 7, since  $f$  is  $(\varepsilon, 1/3)$ -concentrated at  $\varphi(w)$ . Efficiency follows immediately from Lemma 8.  $\square$

*Proof of Theorem 7.* By Corollary 2, there is an algorithm  $\Phi$  for computing  $\varphi(w_i) \pm \varepsilon/c$  with failure probability  $\delta/(2k)$  that runs in time  $O(T' \cdot \log(k/\delta)) + \text{poly}(n, \log k, \log(1/\delta))$  and uses  $m \leq n + O(\log(k/\delta))$  coins. Let  $S$  be the  $(\varepsilon, \delta)$ -steward of Theorem 1 for  $k$  adaptively chosen  $(\varepsilon/c, \delta/(2k))$ -concentrated functions  $f_1, \dots, f_k : \{0, 1\}^m \rightarrow \mathbb{R}$ . (So  $\gamma = \delta/2$ .) When the oracle algorithm makes query  $i$  about string  $w_i$ , let  $f_i(X) = \Phi(w_i, \varepsilon/c, X)$  and give  $f_i$  to  $S$ . When  $S$  outputs a value  $Y_i$ , give it to the oracle algorithm.

Proof of correctness: Each  $f_i$  is  $(\varepsilon/c, \delta/(2k))$ -concentrated at  $\varphi(w_i)$ . Furthermore, each  $f_i$  depends only on the previous oracle responses, i.e.  $Y_1, \dots, Y_{i-1}$ . Therefore, the steward guarantee implies that with probability  $1 - \delta$ , every  $Y_i$  is within  $\pm \varepsilon$  of  $\varphi(w_i)$ . If this occurs, then the oracle algorithm is guaranteed to give a correct output.

Randomness complexity analysis: The number of bits used by the steward is

$$m + O(k + (\log k) \log(1/\delta)) = n + O(k + (\log k) \log(1/\delta)).$$

Runtime analysis: The runtime of the steward is  $\text{poly}(m, k, \log(1/\gamma)) = \text{poly}(n, k, \log(1/\delta))$ . Therefore, the total runtime is bounded by

$$T + k \cdot (O(T' \cdot \log(k/\delta)) + \text{poly}(n, \log k, \log(1/\delta))) + \text{poly}(n, k, \log(1/\delta)),$$

which is bounded by the expression in the theorem statement.  $\square$

## 6.4 The Goldreich-Levin algorithm

**Theorem 8** (Randomness-efficient Goldreich-Levin algorithm). *There is a randomized algorithm that, given oracle access to  $F : \{0, 1\}^n \rightarrow \{-1, 1\}$  and given input parameters  $\delta, \theta > 0$ , outputs a list  $L$  of subsets of  $[n]$  such that with probability  $1 - \delta$ ,*

1. *every  $U$  satisfying  $|\hat{F}(U)| \geq \theta$  is in  $L$ , and*
2. *every  $U \in L$  satisfies  $|\hat{F}(U)| \geq \theta/2$ .*

*The number of queries made by the algorithm is*

$$O\left(\frac{n}{\theta^{11} \log(1/\theta)} \log\left(\frac{n}{\delta\theta}\right)\right),$$

*the number of random bits used by the algorithm is*

$$O(n + (\log n) \log(1/\delta)),$$

*and the runtime of the algorithm is  $\text{poly}(n, 1/\theta, \log(1/\delta))$ .*

For comparison, using standard techniques (the GW sampler, reusing randomness within each round of adaptivity), the Goldreich-Levin algorithm can be implemented in a straightforward way to use  $O(\frac{n}{\theta^6} \log(\frac{n}{\delta\theta}))$  queries and  $O(n^2 + n \log(\frac{n}{\delta\theta}))$  random bits. So our algorithm significantly improves the randomness complexity at the expense of substantially increasing the exponent of  $1/\theta$  in the query complexity.

Toward proving Theorem 8, for a string  $x \in \{0, 1\}^{\leq n}$ , define

$$\mathcal{U}(x) = \{U \subseteq [n] : \forall j \leq |x|, j \in U \iff x_j = 1\}.$$

(That is, we think of  $x \in \{0, 1\}^\ell$  as specifying  $U \cap [\ell]$  in the natural way.) Define  $W_x[F] = \sum_{U \in \mathcal{U}(x)} \hat{F}(U)^2$ . One of the key facts used in the standard Goldreich-Levin algorithm is that  $W_x[F]$  can be estimated using few queries to  $F$ ; here, we use the GW sampler to improve the randomness efficiency of that estimation.

**Lemma 9.** *There is a randomized algorithm that, given oracle access to  $F$  and inputs  $x \in \{0, 1\}^{\leq n}$ ,  $\varepsilon, \delta > 0$ , estimates  $W_x[F]$  to within  $\pm\varepsilon$  with failure probability  $\delta$ . The number of queries is  $O(\log(1/\delta)/\varepsilon^2)$ , the number of random bits is  $O(n + \log(1/\delta))$ , and the runtime is  $\text{poly}(n, 1/\varepsilon, \log(1/\delta))$ .*

*Proof.* Let  $\ell = |x|$ . As shown in the proof of [O'D14, Proposition 3.40],

$$W_x[F] = \mathbb{E}_{\substack{y, y' \in \{0,1\}^\ell \\ z \in \{0,1\}^{n-\ell}}} [F(y, z) \cdot F(y', z) \cdot \chi_x(y) \cdot \chi_x(y')],$$

where  $\chi_x(y) \stackrel{\text{def}}{=} \prod_{j: x_j=1} (-1)^{y_j}$ . Let  $C : \{0, 1\}^{n+\ell} \rightarrow \{0, 1\}$  be the function

$$C(y, y', z) = \frac{1}{2} + \frac{1}{2} \cdot F(y, z) \cdot F(y', z) \cdot \chi_x(y) \cdot \chi_x(y'),$$

so that  $W_x[F] = 2\mathbb{E}_{y, y', z} [C(y, y', z)] - 1$ . We can estimate the expectation of  $C$  to within  $\pm\epsilon/2$  with failure probability  $\delta$  using the GW sampler of Lemma 6, which implies an estimate of  $W_x[F]$  to within  $\pm\epsilon$ . The number of queries made by the GW sampler is  $O(\log(1/\delta)/\epsilon^2)$ , and each query to  $C$  can be evaluated by making 2 queries to  $F$ . The randomness complexity of the GW sampler is  $n + \ell + O(\log(1/\delta))$ , which is  $O(n + \log(1/\delta))$ .  $\square$

The standard Goldreich-Levin algorithm proceeds by finding, for  $\ell = 1$  to  $n$ , the set of all  $x$  with  $|x| = \ell$  such that  $W_x[F] \gtrsim \theta^2$ . In each round, the algorithm estimates  $W_x[F]$  for all strings  $x$  formed by appending a single bit to a string  $x'$  that was previously found to satisfy  $W_{x'}[F] \gtrsim \theta^2$ . This adaptive structure is exactly suited for saving random bits using a steward. To further drive down the randomness complexity, we reduce the number of rounds of adaptivity by appending  $\log(1/\theta)$  bits at a time instead of 1 bit.

*Proof of Theorem 8. Algorithm:*

1. Let  $u = \lfloor \log(1/\theta) \rfloor$ , let  $k = \lceil n/u \rceil$ , and let  $d = \lfloor 2^u \cdot 4/\theta^2 \rfloor$ .
2. Let  $\mathbf{S}$  be a  $(\theta^2/4, \delta)$ -steward for  $k$  adaptively chosen  $(\epsilon, \delta/(2n))$ -concentrated functions  $f_1, \dots, f_k : \{0, 1\}^m \rightarrow \mathbb{R}^d$ , where  $\epsilon \geq \Omega(\theta^2/d)$  and  $m$  will become clear later.
3. Set  $L_0 := \{\text{empty string}\}$ .
4. For  $i = 1$  to  $k$ :
  - (a) If  $|L_{i-1}| > d/2^u$ , abort and output “fail”.
  - (b) Observe that every string in  $L_{i-1}$  has length  $\ell = u(i-1) < n$ . Let  $x_1, \dots, x_t$  be the set of all strings obtained from strings in  $L_{i-1}$  by appending  $\min\{u, n - \ell\}$  bits, so  $t \leq 2^u |L_{i-1}| \leq d$ .
  - (c) Define  $f_i : \{0, 1\}^m \rightarrow \mathbb{R}^t$  by letting  $f_i(X)_j$  be the estimate of  $W_{x_j}[F]$  to within  $\pm\epsilon$  provided by the algorithm of Lemma 9 with failure probability  $\delta/(2dn)$  using randomness  $X$ . Observe that by the union bound,  $f_i$  is  $(\epsilon, \delta/(2n))$ -concentrated at  $(W_{x_1}[F], \dots, W_{x_t}[F])$ .
  - (d) By giving  $f_i$  to  $\mathbf{S}$ , obtain estimates  $\mu_1, \dots, \mu_t$  for  $W_{x_1}[F], \dots, W_{x_t}[F]$ .
  - (e) Set  $L_i := \{x_j : \mu_j \geq \theta^2/2\}$ .
5. Output  $L \stackrel{\text{def}}{=} \bigcup_{x \in L_k} \mathcal{U}(x)$ .

As hopefully became clear,  $m$  is the number of random bits used by the algorithm of Lemma 9. With probability  $1 - \delta$ , all of the responses of  $\mathbf{S}$  are accurate, i.e. every  $\mu_j$  value is within  $\pm\theta^2/4$  of the corresponding  $W_{x_j}[F]$  value. Assume from now on that this has happened.

By the definition of  $L_i$ , every  $x$  in every  $L_i$  satisfies  $W_x[F] \geq \theta^2/4$ . By Parseval's theorem (see e.g. [O'D14, Section 1.4]), this implies that  $|L_i| \leq 4/\theta^2 \leq d/2^u$  for every  $i$ . Therefore, the algorithm does not abort. Let  $\ell_i$  be the length of all the strings in  $L_i$ , so  $\ell_i = ui$  for  $i < k$  and  $\ell_k = n$ . Suppose  $\widehat{F}(U)^2 \geq \theta^2$ . By induction on  $i$ , the unique string  $x \in \{0,1\}^{\ell_i}$  with  $U \in \mathcal{U}(x)$  is placed in  $L_i$ , because the estimate of  $W_x[F]$  is at least  $3\theta^2/4 > \theta^2/2$ . This shows that  $U \in L$ . Conversely, if  $U$  ends up in  $L$ , then the estimate of  $\widehat{F}(U)^2$  in iteration  $i = n$  was at least  $\theta^2/2$ , so  $\widehat{F}(U)^2 \geq \theta^2/4$ . This completes the proof of correctness of the algorithm.

Now, observe that the total number of queries to  $F$  is at most  $kd$  times the  $O(\log(nd/\delta)/\varepsilon^2)$  queries that the algorithm of Lemma 9 makes, i.e. the total number of queries to  $F$  is

$$O\left(\frac{kd^3 \log(nd/\delta)}{\theta^2}\right) = O\left(\frac{n}{\theta^{11} \log(1/\theta)} \log\left(\frac{n}{\delta\theta}\right)\right).$$

The randomness complexity of the algorithm is just the randomness complexity of  $S$ . We will use the steward of Theorem 1 with  $\gamma = \delta/2$ , so the randomness complexity is  $m + O(k \log(d+1) + (\log k) \log(1/\delta))$ . Since  $m \leq O(n + \log(n/(\delta\theta)))$ , the total randomness complexity is

$$O\left(n + \frac{n}{\log(1/\theta)} \log(1/\theta) + (\log n) \log(1/\delta) + \log(1/\theta)\right) = O(n + (\log n) \log(1/\delta) + \log(1/\theta)).$$

To get rid of the  $\log(1/\theta)$  term as claimed in the theorem statement, just notice that we can assume without loss of generality that  $\theta \geq 2^{-n+1}$ , because any nonzero Fourier coefficient of a  $\{-1,1\}$ -valued function has absolute value at least  $2^{-n+1}$ . The total runtime of the algorithm is clearly  $\text{poly}(n, 1/\theta, \log(1/\delta))$ .  $\square$

## 7 Randomness complexity lower bound

To understand the following lemma, imagine the perspective of  $O$  after  $i-1$  rounds of  $O \leftrightarrow S(Z)$ , where  $Z$  was chosen uniformly at random from  $\{0,1\}^m$ . Let  $R$  be the set of  $z$  such that the hypothesis that  $S$  is using randomness string  $z$  is compatible with everything that  $O$  has seen so far. Then at this point,  $O$ 's posterior distribution for  $Z$  is uniform over  $R$ . The following lemma says that with respect to this posterior distribution,  $O$  can choose  $f_i$  such that either  $O$  will learn  $\Omega(1)$  bits of information about  $Z$  based on  $Y_i$ , or else  $S$  will have a failure probability of  $\Omega(1)$  in round  $i$ .

**Lemma 10.** *Suppose  $S$  is an  $m$ -coin  $(\varepsilon', \delta')$ -steward for  $k$  adaptively chosen  $(\varepsilon, \delta)$ -concentrated functions  $f_1, \dots, f_k : \{0,1\}^n \rightarrow \mathbb{R}$  and  $O$  is a deterministic owner. Fix  $i \in [k]$ . For a function  $g : \{0,1\}^n \rightarrow \mathbb{R}$ , let  $O[g]$  be the owner that simulates  $O$  for rounds  $1, 2, \dots, i-1$ , but chooses  $g$  in round  $i$  regardless of what  $O$  would have chosen. Let  $R \subseteq \{0,1\}^m$  be a nonempty set such that the transcript of the first  $i-1$  rounds of  $O \leftrightarrow S(Z)$  is the same for every  $Z \in R$ . Assume  $\delta \geq 2^{-n}$ . Then there exists  $g$  that is  $(\varepsilon, \delta)$ -concentrated at  $\mu$  such that either*

1.  $\max_{y \in \mathbb{R}} \Pr_{Z \in R}[Y_i = y \text{ in } O[g] \leftrightarrow S(Z)] \leq 0.8$ , or
2.  $\Pr_{Z \in R}[|Y_i - \mu_i| > \varepsilon' \text{ in } O[g] \leftrightarrow S(Z)] \geq 0.2$ .

*Proof.* For each  $j \in \mathbb{Z}$ , let  $g_j : \{0,1\}^n \rightarrow \mathbb{R}$  be constant at  $\varepsilon j$ . If some  $g_j$  satisfies Condition 1, we're done. So assume that for each  $g_j$ , there is some  $y_j \in \mathbb{R}$  such that  $\Pr_{Z \in R}[Y_j = y_j \text{ in } O[g] \leftrightarrow S(Z)] > 0.8$ . If  $y_j$  does not depend on  $j$ , then since  $0.2 < 0.8$ , there is some  $g_j$  that satisfies Condition 2, so we are again done. Therefore, assume there is some  $j$  such that  $y_j \neq y_{j+1}$ .

Define  $q : R \rightarrow \{0, 1\}^n$  by letting  $q(Z) =$  the value  $X_i$  chosen by  $\mathbf{S}$  in  $\mathbf{O} \leftrightarrow \mathbf{S}(Z)$ . First, assume there is some  $x^*$  such that  $\Pr_{Z \in R}[q(Z) = x^*] \geq 0.4$ . For  $s \in \{\pm 1\}$ , define  $g^s : \{0, 1\}^n \rightarrow \mathbb{R}$  by

$$g^s(x) = \begin{cases} 0 & \text{if } x = x^* \\ s \cdot 2\varepsilon' & \text{otherwise.} \end{cases}$$

Then  $g^s(x)$  is  $(0, 2^{-n})$ -concentrated at  $s \cdot 2\varepsilon'$ . Let  $\mathbf{O}'$  be the randomized owner that tosses a coin to decide whether to simulate  $\mathbf{O}[g^{+1}]$  or  $\mathbf{O}[g^{-1}]$ . Then when  $Z \in R$  is chosen uniformly at random, in  $\mathbf{O}' \leftrightarrow \mathbf{S}(Z)$ , there is a 0.4 chance that  $f_i(X_i) = 0$ , in which case  $\mathbf{S}(Z)$  has only a 50% chance of correctly guessing  $s$ . This shows that  $\Pr_{Z \in R}[|Y_i - \mu_i| > \varepsilon' \text{ in } \mathbf{O}' \leftrightarrow \mathbf{S}(Z)] \geq 0.2$ , and hence either  $g^{+1}$  or  $g^{-1}$  satisfies Condition 2, so we are again done. Therefore, assume that for every  $x^*$ ,  $\Pr_{Z \in R}[q(Z) = x^*] < 0.4$ .

For  $t \in \{j, j+1\}$ , let

$$A_t = \{Z \in R : Y_i = y_t \text{ in } \mathbf{O}[g_t] \leftrightarrow \mathbf{S}(Z)\},$$

so that  $|A_t| > 0.8|R|$ . We define  $g$  by the following greedy algorithm. Two players, which we identify with  $A_j$  and  $A_{j+1}$ , alternate taking turns. When it is  $A_t$ 's turn, she finds the string  $x \in \{0, 1\}^n$  such that  $g(x)$  is not yet defined that maximizes  $q^{-1}(x) \cap A_t$ , and defines  $g(x) = \varepsilon t$ . This continues for  $2^n$  turns until  $g$  is defined everywhere.

Clearly,  $g$  thus defined is  $(\varepsilon, 0)$ -concentrated. We will show that  $g$  satisfies Condition 1. Proof: Say  $z \in \{0, 1\}^m$  is *good* for  $A_t$  if  $z \in A_t$  and  $g(q(z)) = \varepsilon t$ . In these terms, on  $A_t$ 's turn, she defines  $g$  on one more point in order to maximize the number of  $z$  that become good for  $A_t$ . Say that  $z \in \{0, 1\}^m$  is *bad* for  $A_t$  if  $z \in A_t$  and  $g(q(z)) \neq \varepsilon t$ . When it is not  $A_t$ 's turn, some  $z$  may become bad for  $A_t$ , but the crucial point is that the number of  $z$  that become bad for  $A_t$  is *at most* the number of  $z$  that became *good* for  $A_t$  in the *previous* turn (simply because of the greedy choice that  $A_t$  made in the previous turn.) This would show that half of  $A_t$  is good for  $A_t$ , except for one annoyance: the first turn, where some  $z$  become bad for the second player with no corresponding previous turn, when  $z$  became good. But we already showed that for every  $x$ ,  $|q^{-1}(x)| \leq 0.4|R|$ , so the first turn does not matter too much: at the end of the construction, for each  $t$ , the number of  $z$  that are good for  $A_t$  is at least

$$\frac{1}{2}(|A_t| - 0.4|R|) \geq \frac{1}{2}(0.8|R| - 0.4|R|) = 0.2|R|.$$

By construction, if  $z$  is good for  $A_t$ , then  $Y_i = y_t$  in  $\mathbf{O}[g] \leftrightarrow \mathbf{S}(z)$ . Therefore, for each  $t$ ,  $\Pr_{Z \in R}[Y_i = y_t \text{ in } \mathbf{O}[g] \leftrightarrow \mathbf{S}(Z)] \geq 0.2$ , which implies Condition 1 since  $y_j \neq y_{j+1}$ .  $\square$

Having proved Lemma 10, we are ready to prove our randomness complexity lower bound. The idea is that  $\mathbf{O}$  will spend the first  $k - 1$  rounds learning as much information as possible about  $\mathbf{S}$ 's randomness string using Lemma 10 (unless she gets lucky and is able to cause  $\mathbf{S}$  to have an  $\Omega(1)$  failure probability in one of these rounds, in which case she will take the opportunity.) Then, in round  $k$ ,  $\mathbf{O}$  uses everything she's learned about  $\mathbf{S}$ 's randomness string to choose  $f_k$  so as to maximize  $\mathbf{S}$ 's failure probability in that round.

**Theorem 9.** *Suppose  $\mathbf{S}$  is an  $m$ -coin  $(\varepsilon', \delta')$ -steward for  $k$  adaptively chosen  $(\varepsilon, \delta)$ -concentrated functions  $f_1, \dots, f_k : \{0, 1\}^n \rightarrow \mathbb{R}^d$ . Assume  $\delta' < 0.2$  and  $\delta \geq 2^{-n}$ . Then  $m \geq n + \Omega(k) - \log_2(\delta'/\delta)$ .*

*Proof.* Without loss of generality, assume  $d = 1$ . Let  $\mathbf{O}$  be the following owner:

1. For  $i = 1$  to  $k$ :

- (a) Let  $y_1, y_2, \dots, y_{i-1}$  be the responses received so far.
- (b) Let  $R \subseteq \{0, 1\}^m$  be the set of  $z$  such that in  $O \leftrightarrow S(z)$ ,  $Y_j = y_j$  for every  $j < i$ . (By induction, we have already defined the behavior of  $O$  in rounds  $1, 2, \dots, i-1$ , so  $R$  is well-defined. In other words,  $R$  is the set of  $z$  that are compatible with what  $O$  has seen so far.)
- (c) If  $i < k$ :
  - i. Choose  $f_i = g$ , where  $g$  is the function guaranteed by Lemma 10. (Again,  $O$  is already defined and deterministic for rounds  $1, 2, \dots, i-1$ , so we can sensibly apply the lemma.)
- (d) Otherwise, if  $i = k$ :
  - i. Pick  $S \subseteq R$ ,  $|S| = \min\{\lceil \delta 2^n \rceil, |R|\}$  uniformly at random, pick  $s \in \{\pm 1\}$  independently and uniformly at random, and choose

$$f_k(x) = \begin{cases} 0 & \text{if } x \in q(S) \\ s \cdot 2\varepsilon' & \text{otherwise.} \end{cases}$$

(Note that  $f_k$  is  $(0, \delta)$ -concentrated at  $s \cdot 2\varepsilon'$ , because  $|q(S)| \leq |S| \leq \delta 2^n$ .)

To analyze  $O$ , in  $O \leftrightarrow S$ , say that  $O$  *tries to win* in round  $i$  if either  $i = k$  or else  $i < k$  and the function  $f_i$  chosen satisfies Condition 2 in Lemma 10. For a string  $z \in \{0, 1\}^m$ , let  $w(z) \in [k]$  be the index of the first round in which  $O$  tries to win in  $O \leftrightarrow S(z)$ , and let  $\tau(z)$  be the transcript of rounds  $1, 2, \dots, w(z) - 1$  in  $O \leftrightarrow S(z)$ . Note that since  $O$  is deterministic in rounds  $1, 2, \dots, k-1$ ,  $w(z)$  and  $\tau(z)$  are not random variables. Define an equivalence relation on  $\{0, 1\}^m$  by saying that  $z \sim z'$  if and only if  $\tau(z) = \tau(z')$ . Say  $O$  uses  $v$  random bits. We first show that for each equivalence class  $\bar{z}$ ,

$$\Pr_{Z \in \bar{z}, V \in \{0,1\}^v} [\|Y_{w(\bar{z})} - \mu_{w(\bar{z})}\|_\infty > \varepsilon' \text{ in } O(V) \leftrightarrow S(Z)] \geq \min\{0.2, \delta \cdot (1/0.8)^{k-1} \cdot 2^{n-m-2}\}. \quad (1)$$

Proof: Observe that in round  $w(\bar{z})$ ,  $O$ 's set  $R$  is precisely  $\bar{z}$ . If  $w(\bar{z}) < k$ , then Condition 2 of Lemma 10 immediately implies that the failure probability in Equation 1 is at least 0.2. Suppose instead that  $w(\bar{z}) = k$ . Then in every previous round,  $O$  did not try to win, i.e.  $O$  chose a function satisfying Condition 1 of Lemma 10. This implies that in every previous round,  $O$ 's set  $R$  decreased in size by a factor of 0.8. So at the beginning of round  $k$ ,  $|R| \leq 0.8^{k-1} \cdot 2^m$ . The probability (over  $Z \in \bar{z}$ ) that  $S$  chooses  $X_k$  such that  $f_k(X_k) = 0$  is

$$\begin{aligned} \frac{|S|}{|R|} &= \frac{\min\{\lceil \delta 2^n \rceil, |R|\}}{|R|} \\ &\geq \min\left\{1, \delta 2^{n-m-1} (1/0.8)^{k-1}\right\}. \end{aligned}$$

Conditioned on  $f_k(X_k) = 0$ , the probability of the event in Equation 1 is at least 0.5, because conditioned on  $f_k(X_k) = 0$ ,  $s$  is independent of everything  $S$  has seen. Therefore, the probability of the event in Equation 1 is at least  $\min\{0.5, \delta 2^{n-m-2} (1/0.8)^{k-1}\}$ , completing the proof of Equation 1.

Now, to prove the theorem, observe that

$$\begin{aligned}
\delta' &\geq \Pr_{Z \in \{0,1\}^m, V \in \{0,1\}^v} [\max_i \|\mu_i - Y_i\|_\infty > \varepsilon' \text{ in } \mathcal{O}(V) \leftrightarrow \mathcal{S}(Z)] \\
&\geq \Pr_{Z \in \{0,1\}^m, V \in \{0,1\}^v} [\|\mu_{w(Z)} - Y_{w(Z)}\|_\infty > \varepsilon' \text{ in } \mathcal{O}(V) \leftrightarrow \mathcal{S}(Z)] \\
&= \sum_{\bar{z}} \Pr_{Z \in \{0,1\}^m} [Z \in \bar{z}] \cdot \Pr_{Z' \in \bar{z}, V \in \{0,1\}^v} [\|\mu_{w(\bar{z})} - Y_{w(\bar{z})}\|_\infty > \varepsilon' \text{ in } \mathcal{O}(V) \leftrightarrow \mathcal{S}(Z')] \\
&\geq \sum_{\bar{z}} \Pr_{Z \in \{0,1\}^m} [Z \in \bar{z}] \cdot \min\{0.2, \delta \cdot (1/0.8)^{k-1} 2^{n-m-2}\} \\
&= \min\{0.2, \delta \cdot (1/0.8)^{k-1} 2^{n-m-2}\}.
\end{aligned}$$

We assumed that  $\delta' < 0.2$ , so we can conclude that  $\delta' \geq \delta \cdot (1/0.8)^{k-1} 2^{n-m-2}$ . Rearranging proves that

$$\begin{aligned}
m &\geq (n-2) + (k-1) \log_2(1/0.8) - \log_2(\delta'/\delta) \\
&\geq n + \Omega(k) - \log_2(\delta'/\delta),
\end{aligned}$$

completing the proof. □

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## A Generalized shifting and rounding algorithm

In this section, we show how to generalize the steward  $\mathbf{S}_0$  to achieve a tradeoff between its error and the branching factor of the certification tree  $T_{\mathbf{O}}$ . Fix any factorization of  $d$  into  $d = d_0 d_1$ . Partition  $[d]$  as  $[d] = J_1 \cup J_2 \cup \dots \cup J_{d_1}$ , where  $|J_t| = d_0$  for each  $t$ . Instead of partitioning  $\mathbb{R}$  into intervals of length  $2(d+1)\varepsilon$ , partition  $\mathbb{R}$  into intervals of length  $2(d_0+1)\varepsilon$ . The following algorithm for computing  $Y_i$  from  $W_i$  generalizes that of Section 2.1:

1. For each  $t \in [d_1]$ :
  - (a) Find  $\Delta_{it} \in [d_0+1]$  such that for every  $j \in J_t$ , there is a single interval that entirely contains  $[W_{ij} + (2\Delta_{it} - 1)\varepsilon, W_{ij} + (2\Delta_{it} + 1)\varepsilon]$ . (Such a  $\Delta_{it}$  exists by Lemma 1.)
  - (b) For every  $j \in J_t$ , set  $Y_{ij} = \text{Round}(W_{ij} + 2\Delta_{it}\varepsilon)$ .

The following lemma is the appropriate generalization of Lemma 2:

**Lemma 11.** *Assume  $\delta < 1/2$ . Let  $\Sigma = [d_0+1]^{d_1} \cup \{\perp\}$ . For any deterministic owner  $\mathbf{O}$ , there exists a  $(k, n, \Sigma)$  block decision tree  $T_{\mathbf{O}}$  with the following properties.*

1. *For any internal node  $v$ ,  $\Pr_{X \in \{0,1\}^n}[v(X) = \perp] \leq \delta$ .*
2. *Fix  $X_1, \dots, X_k \in \{0,1\}^n$ , and suppose that the path from the root to  $T_{\mathbf{O}}(X_1, \dots, X_k)$  does not include any  $\perp$  nodes. Then in  $\mathbf{O} \leftrightarrow \mathbf{S}_0(X_1, \dots, X_k)$ ,  $\max_i \|Y_i - \mu_i\|_\infty \leq O(d_0\varepsilon)$ .*

The proof of Lemma 11 is essentially the same as the proof of Lemma 2; we record the details below.

*Proof of Lemma 11.* Consider a thought experiment in which  $S_0$  interacts with  $O$  through two mediators  $M_1$  and  $M_2$  as follows:

1. For  $i = 1$  to  $k$ :

- (a)  $O$  chooses  $f_i : \{0, 1\}^n \rightarrow \mathbb{R}^d$  that is  $(\varepsilon, \delta)$ -concentrated at some point  $\mu_i \in \mathbb{R}^d$  and gives it to  $M_1$ , who gives it to  $M_2$ .
- (b)  $M_1$  and  $M_2$  both compute the lexicographically smallest  $\hat{\mu}_i \in \mathbb{R}^d$  such that  $f_i$  is  $(\varepsilon, \delta)$ -concentrated at  $\hat{\mu}_i$ .
- (c)  $S_0$  chooses a string  $X_i \in \{0, 1\}^n$  and gives it to  $M_2$ , who gives  $f_i(X_i) \in \mathbb{R}^d$  to  $S_0$ .
- (d)  $S_0$  chooses  $Y_i \in \mathbb{R}^d$  and gives it to  $M_2$ .
- (e) Say a vector  $(\Delta_1, \dots, \Delta_{d_1})$  is *compatible* with  $Y_i$  if  $Y_{ij} = \text{Round}(\hat{\mu}_{ij} + 2\Delta_t\varepsilon)$  for every  $t \in [d_1]$  and every  $j \in J_t$ .  $M_2$  computes

$$\hat{\Delta}_i = \begin{cases} \text{the first } (\Delta_1, \dots, \Delta_{d_1}) \in [d_0 + 1]^{d_1} \text{ compatible with } Y_i & \text{if any exist} \\ \perp & \text{otherwise} \end{cases}$$

and gives it to  $M_1$ .

- (f)  $M_1$  computes  $\hat{Y}_i = (\hat{Y}_{i1}, \dots, \hat{Y}_{id})$ , where for each  $t \in [d_1]$  and each  $j \in J_t$ ,

$$\hat{Y}_{ij} = \begin{cases} \text{Round}(\hat{\mu}_{ij} + 2\hat{\Delta}_{it}\varepsilon) & \text{if } \hat{\Delta}_i \neq \perp \\ 0 & \text{otherwise} \end{cases}$$

and gives  $\hat{Y}_i$  to  $O$ .

The definition of  $T_O$  is exactly the same as in the proof of Lemma 2, except that the notation  $O \xleftrightarrow{2} S_0$  now refers to the above modified two-mediator thought experiment. To prove Condition 1 in the lemma statement, we must show that in each round of  $O \xleftrightarrow{2} S_0$ ,  $\Pr[\hat{\Delta}_i = \perp] \leq \delta$ . Indeed, by concentration, with probability  $1 - \delta$ , for every  $j$ ,  $|W_{ij} - \hat{\mu}_{ij}| \leq \varepsilon$ . In this case, by the construction of  $S_0$ ,  $W_{ij} + 2\Delta_{it}\varepsilon$  and  $\hat{\mu}_{ij} + 2\Delta_{it}\varepsilon$  are in the same interval for every  $t \in [d_1]$  and every  $j \in J_t$ . Therefore, in this case, there is at least one vector  $(\Delta_1, \dots, \Delta_{d_1})$  compatible with  $Y_i$ , namely the vector of  $\Delta_{it}$  values used by  $S_0$ . To prove Condition 2 in the lemma statement, suppose the path from the root node to  $T_O(X_1, \dots, X_k)$  does not include any  $\perp$  nodes. Then in  $O \xleftrightarrow{2} S_0(X_1, \dots, X_k)$ , for every  $i$ ,  $\hat{\Delta}_i \neq \perp$ . This implies that every  $Y_{ij}$  is of the form  $\text{Round}(\hat{\mu}_{ij} + 2\hat{\Delta}_{it}\varepsilon)$  for some  $\hat{\Delta}_{it} \in [d_0 + 1]$ . Therefore,  $|Y_{ij} - \hat{\mu}_{ij}| \leq 3(d_0 + 1)\varepsilon$ , since  $2\hat{\Delta}_{it}\varepsilon \leq 2(d_0 + 1)\varepsilon$  and rounding introduces at most  $(d_0 + 1)\varepsilon$  additional error. Since  $\delta < 1/2$ ,  $\|\hat{\mu}_i - \mu_i\|_\infty \leq 2\varepsilon$ , so by the triangle inequality, for every  $i$ ,  $\|Y_i - \mu_i\|_\infty \leq 3(d_0 + 1)\varepsilon + 2\varepsilon = (3d_0 + 5)\varepsilon$ . Just as in the proof of Lemma 2, the same bound holds in  $O \leftrightarrow S_0$ .  $\square$

## B The Saks-Zhou steward

In this section, for completeness, we give the description and analysis of the Saks-Zhou steward. This algorithm and analysis are the same in spirit as what appears in [SZ99], but the presentation has been changed to match our framework. None of our results use this steward, but it is interesting to see how the stewards compare.

**Proposition 1.** *For any  $n, k, d \in \mathbb{N}$  and any  $\varepsilon, \delta, \gamma > 0$ , there exists an  $(O(kd\varepsilon/\gamma), k\delta + \gamma)$ -steward for  $k$  adaptively chosen  $(\varepsilon, \delta)$ -concentrated functions  $f_1, \dots, f_k : \{0, 1\}^n \rightarrow \mathbb{R}^d$  with randomness complexity*

$$n + O(k \log k + k \log d + k \log(1/\gamma)).$$

*The total running time of the steward is  $\text{poly}(n, k, d, \log(1/\varepsilon), \log(1/\gamma))$ .*

*Proof.* Let  $u$  be the smallest power of two such that  $u \geq 2kd/\gamma$ . (The only reason we choose a power of two is so that we can cleanly draw a uniform random element of  $[u]$  using  $\log u$  random bits.) Partition  $\mathbb{R}$  into half-open intervals of length  $\ell = u\varepsilon$ . For  $w \in \mathbb{R}$ , let  $\text{Round}(w)$  be the midpoint of the interval containing  $w$ . Algorithm S:

1. Pick  $X \in \{0, 1\}^n$  uniformly at random *once*.
2. For  $i = 1$  to  $k$ :
  - (a) Obtain  $W_i = f_i(X) \in \mathbb{R}^d$ .
  - (b) Pick  $\Delta_i \in [u]$  uniformly at random.
  - (c) Return  $Y_i = (Y_{i1}, \dots, Y_{id})$ , where  $Y_{ij} = \text{Round}(W_i + \Delta_i \varepsilon)$ .

Proof of correctness: Fix any deterministic owner  $O$ . Consider a thought experiment in which  $O$  interacts with a party  $P$  who, from the perspective of  $O$ , seems to act like the mediator  $M$ . (What  $O$  does not realize is that there is no steward in this thought experiment.) This party  $P$  behaves as follows:

1. For  $i = 1$  to  $k$ :
  - (a) Pick  $\Delta_i \in [u]$  uniformly at random.
  - (b) Obtain  $f_i$ , and find some point  $\hat{\mu}_i \in \mathbb{R}^d$  where  $f_i$  is  $(\varepsilon, \delta)$ -concentrated.
  - (c) Return  $Y_i = (Y_{i1}, \dots, Y_{id})$ , where  $Y_{ij} = \text{Round}(\hat{\mu}_{ij} + \Delta_i \varepsilon)$ .

Write  $O \leftrightarrow P$  to denote the above interaction. For a vector  $\vec{\Delta} = (\Delta_1, \dots, \Delta_k) \in [u]^k$ , let  $f_1^{[\vec{\Delta}]}, \dots, f_k^{[\vec{\Delta}]}$  be the functions that  $O$  chooses in  $O \leftrightarrow P(\vec{\Delta})$ , and let  $\hat{\mu}_i^{[\vec{\Delta}]}$  be the point at which  $f_i^{[\vec{\Delta}]}$  is concentrated that  $P$  chooses in  $O \leftrightarrow P(\vec{\Delta})$ . Observe that

$$\Pr_{\substack{\vec{\Delta} \in [u]^k \\ X \in \{0, 1\}^n}} [\text{for some } i, \|f_i^{[\vec{\Delta}]}(X) - \hat{\mu}_i^{[\vec{\Delta}]}\|_\infty > \varepsilon] \leq k\delta. \quad (2)$$

(Imagine picking  $\vec{\Delta}$  first, and then apply the union bound over the  $k$  different values of  $i$ .) Next, observe that

$$\Pr_{\substack{\vec{\Delta} \in [u]^k \\ X \in \{0, 1\}^n}} [\text{for some } i, j, [\hat{\mu}_{ij}^{[\vec{\Delta}]} + (\Delta_i - 1)\varepsilon, \hat{\mu}_{ij}^{[\vec{\Delta}]} + (\Delta_i + 1)\varepsilon] \text{ is not entirely contained in one interval}] \leq \gamma. \quad (3)$$

(Indeed, for each  $i, j$ , the probability is just  $2/u$ , so by the union bound, the probability is at most  $2kd/u \leq \gamma$ .) Now, by the union bound, assume from now on that  $\vec{\Delta}, X$  are such that neither the event of Equation 2 nor the event of Equation 3 takes place. Assume without loss of generality that  $\delta < 1/2$ . We will show that in  $O \leftrightarrow S(X, \vec{\Delta})$ , for every  $i$ ,  $\|Y_i - \mu_i\|_\infty \leq 1.5\ell + 3\varepsilon$ .

We first show by induction on  $i$  that in  $O \leftrightarrow S(X, \vec{\Delta})$ , every  $f_i$  is precisely  $f_i^{[\vec{\Delta}]}$ . In the base case  $i = 1$ , this is trivial. For the inductive step, since the bad event of Equation 2 did not occur, we

know that  $f_i(X)$  is  $\hat{\mu}_i^{[\vec{\Delta}]} \pm \varepsilon$ . Therefore, since the bad event of Equation 3 did not occur, for every  $j$ ,  $\text{Round}(f_{ij}(X) + \Delta_i \varepsilon) = \text{Round}(\hat{\mu}_{ij}^{[\vec{\Delta}]} + \Delta_i \varepsilon)$ . Therefore, the value  $Y_i$  in  $\mathcal{O} \leftrightarrow \mathcal{S}(X, \vec{\Delta})$  is the same as the value  $Y_i$  in  $\mathcal{O} \leftrightarrow \mathcal{P}(\vec{\Delta})$ , and hence  $\mathcal{O}$  chooses the same  $f_{i+1}$  in both cases. This completes the induction.

Again using the fact that the bad event of Equation 2 did not occur, this immediately implies that in  $\mathcal{O} \leftrightarrow \mathcal{S}(X, \vec{\Delta})$ , every  $f_i(X)$  is within  $\ell_\infty$  distance  $\varepsilon$  of a point where  $f_i$  is  $(\varepsilon, \delta)$ -concentrated. Since  $\delta < 1/2$ , this implies that every  $f_i(X)$  is within  $\ell_\infty$  distance  $3\varepsilon$  of  $\mu_i$ . Shifting by  $\Delta_i \varepsilon$  and rounding introduce at most  $1.5\ell$  additional error, showing that  $\|Y_i - \mu_i\| \leq 1.5\ell + 3\varepsilon$  as claimed. To complete the proof of correctness, note that  $1.5\ell + 3\varepsilon \leq O(kd\varepsilon/\gamma)$ .

The randomness complexity of this steward is  $n$  bits (for  $X$ ) plus the randomness needed for  $\vec{\Delta}$ , for a total randomness complexity of

$$n + k \log u \leq n + O(k \log k + k \log d + k \log(1/\gamma)).$$

The steward clearly runs in  $\text{poly}(n, k, d, \log(1/\varepsilon), \log(1/\gamma))$  time.  $\square$

## C Nonconstructive PRG for block decision trees

For completeness, we record the details of the standard nonconstructive argument that there exists a PRG for block decision trees with a small seed length.

**Lemma 12.** *Suppose  $\mathcal{C}$  is a class of Boolean functions  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  such that a function in  $\mathcal{C}$  can be specified using  $t$  bits, i.e.  $|\mathcal{C}| \leq 2^t$ . Then for any  $\gamma$ , there exists a  $\gamma$ -PRG  $\text{Gen} : \{0, 1\}^s \rightarrow \{0, 1\}^n$  for  $\mathcal{C}$  with seed length*

$$s \leq \log t + 2 \log(1/\gamma) + O(1).$$

*Proof.* Consider picking  $\text{Gen}$  uniformly at random from the set of all functions  $\{0, 1\}^s \rightarrow \{0, 1\}^n$ . Fix  $C \in \mathcal{C}$ , and let  $\mu(C) = \Pr_x[C(x) = 1]$ . Then for each fixed seed  $x \in \{0, 1\}^s$ , the probability (over  $\text{Gen}$ ) that  $C(\text{Gen}(x)) = 1$  is precisely  $\mu(C)$ . Therefore, the expected fraction of  $x$  such that  $C(\text{Gen}(x)) = 1$  is precisely  $\mu(C)$ , and by Hoeffding's inequality,

$$\Pr_{\text{Gen}} \left[ \left| \frac{\#\{x : C(\text{Gen}(x)) = 1\}}{2^s} - \mu(C) \right| > \gamma \right] \leq 2^{-\Omega(\gamma^2 2^s)}.$$

Therefore, by the union bound, the probability that the above bad event holds for *any*  $C$  is at most  $2^{t-\Omega(\gamma^2 2^s)}$ . If we choose  $s$  large enough, this probability will be less than 1, showing that there exists a  $\text{Gen}$  that works for all  $C$ . How large do we need to choose  $s$ ? There is some constant  $c$  such that it is sufficient to have  $c\gamma^2 2^s > t$ . Taking logarithms completes the proof.  $\square$

**Proposition 2.** *For any  $k, n \in \mathbb{N}$ , any finite alphabet  $\Sigma$ , and any  $\gamma > 0$ , there exists a  $\gamma$ -PRG  $\text{Gen} : \{0, 1\}^s \rightarrow \{0, 1\}^{nk}$  for  $(k, n, \Sigma)$  block decision trees with seed length*

$$s \leq n + k \log |\Sigma| + 2 \log(1/\gamma) + O(1).$$

*Proof.* Let  $\mathcal{C}$  be the class of all Boolean functions  $f : \{0, 1\}^{nk} \rightarrow \{0, 1\}$  of the form  $f(x) = g(T(x))$ , where  $T$  is a  $(k, n, \Sigma)$  block decision tree. To specify a function  $f \in \mathcal{C}$ , we need to specify (1) a bit for each leaf of  $T$  and (2) a function  $v : \{0, 1\}^n \rightarrow \Sigma$  for each internal node of  $T$ . In total, this

number of bits  $t$  is

$$\begin{aligned}
t &= |\Sigma|^k + 2^n \lceil \log |\Sigma| \rceil \cdot \sum_{i=0}^{k-1} |\Sigma|^i \\
&\leq |\Sigma|^k + 2^{n+1} \log |\Sigma| \cdot \frac{|\Sigma|^k - 1}{|\Sigma| - 1} \\
&\leq |\Sigma|^k + 2^{n+1} |\Sigma|^k \\
&\leq 2^{n+2} |\Sigma|^k.
\end{aligned}$$

By Lemma 12, this implies that there is a  $\gamma$ -PRG  $\text{Gen} : \{0, 1\}^s \rightarrow \{0, 1\}^{nk}$  for  $\mathcal{C}$  with seed length  $n + k \log |\Sigma| + 2 \log(1/\gamma) + O(1)$ . The “operational” characterization of total variation distance implies that  $\text{Gen}$  is also a  $\gamma$ -PRG for  $(k, n, \Sigma)$  block decision trees as defined in Section 3.  $\square$